Complex Variables Late Summer 2021 Solutions

Q1

(a) Let $z = x+iy$. Then $z\bar{z} = x^2+y^2$ and we have

$$5(x+iy) - 3i = 4x^2+4y^2 \Leftrightarrow 5x + i5y - 3i - 4x^2 - 4y^2 = 0$$

$$\Leftrightarrow \quad 5x - 4x^2 - 4y^2 + i(5y-3) = 0$$

Since for all $z_1, z_2 \in \mathbb{C}$ $z_1 = z_2 \Leftrightarrow \operatorname{Re} z_1 = \operatorname{Re} z_2$ and $\operatorname{Im} z_1 = \operatorname{Im} z_2$ we obtain the following system of equations

$$\begin{cases} 1)\ 5x - 4x^2 - 4y^2 = 0 \\ 2)\ 5y - 3 = 0 \end{cases}$$

From 2) $y = 3/5$, therefore we obtain from 1

$$5x - 4x^2 - \frac{36}{25} = 0 \Leftrightarrow 4x^2 - 5x + \frac{36}{25} = 0$$

$$\Leftrightarrow \quad x^2 - \frac{5}{4}x + \frac{36}{100} = 0$$

$$x_{1,2} = \frac{\frac{5}{4} \pm \sqrt{\frac{25}{16} - 4\,\frac{36}{100}}}{2} = \frac{\frac{5}{4} \pm \sqrt{\frac{481}{400}}}{2} = \frac{5}{8} \pm \frac{\sqrt{481}}{40}$$

Thus, there are 2 solutions

$$x = \frac{5}{8} + \frac{\sqrt{481}}{40} \quad y = 3/5$$

$$x = \frac{5}{8} - \frac{\sqrt{481}}{40} \quad y = 3/5,$$

namely

$$z_1 = \frac{5}{8} + \frac{\sqrt{481}}{40} + i\,3/5$$

$$z_2 = \frac{5}{8} - \frac{\sqrt{481}}{40} + i\,3/5$$

(b) $z^7 = -128 \iff z = \sqrt[7]{-128}$. Let us write -128 in polar form

$|-128| = 128 \quad \cos\theta = \frac{-128}{128} = -1, \ \sin\theta = \frac{0}{128} = 0,$

thus $\theta = \pi + 2\pi k$

Thus

$$\sqrt[7]{-128} = \sqrt[7]{128}\left(\cos\left(\frac{\pi + 2\pi k}{7}\right) + i\sin\left(\frac{\pi + 2\pi k}{7}\right)\right)$$

for $k = 0, 1, 2, 3, 4, 5, 6.$

Since $\sqrt[7]{128} = 2$

$k=0 \quad 2\left(\cos\frac{\pi}{7} + i\sin\frac{\pi}{7}\right)$

$k=1 \quad 2\left(\cos\frac{3\pi}{7} + i\sin\frac{3\pi}{7}\right)$

$k=2 \quad 2\left(\cos\frac{5\pi}{7} + i\sin\frac{5\pi}{7}\right)$

$k=3 \quad 2\left(\cos\frac{7\pi}{7} + i\sin\frac{7\pi}{8}\right) = -2$

$k=4 \quad 2\left(\cos\frac{9\pi}{7} + i\sin\frac{9\pi}{7}\right)$

$k=5 \quad 2\left(\cos\frac{11\pi}{7} + i\sin\frac{11\pi}{7}\right)$

$k=6 \quad 2\left(\cos\frac{13\pi}{7} + i\sin\frac{13\pi}{7}\right)$

Q2

(a) Let $z = x+iy$, $f(z) = f(x,y) = u(x,y)+iv(x,y)$. Then $\operatorname{Re} z = x$, $\operatorname{Im} f(z) = v(x,y)$, namely we have $3x = 2v(x,y)$. We are looking for all the analytic functions of the form

$$f(x,y) = u(x,y) + i\,\tfrac{3}{2}x$$

Since $f(x,y)$ is analytic, u and v are differentiable and CR equations hold:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Since $\frac{\partial v}{\partial y} = 0$ we conclude that $\frac{\partial u}{\partial x} = 0$. Since $\frac{\partial v}{\partial x} = \frac{3}{2}$, we conclude that $\frac{\partial u}{\partial y} = -\frac{3}{2}$

Integration with respect to y of $\frac{\partial u}{\partial y}$ gives

$$u(x,y) = \int \frac{\partial u}{\partial y}\,dy = \int -\tfrac{3}{2}\,dy = -\tfrac{3}{2}y + c(x)$$

To find $c(x)$ we differentiate $u(x,y)$ with respect to x and compare to $\frac{\partial u}{\partial x}$ obtained from CR equations

$$\frac{\partial u}{\partial x} = c'(x) = 0 \Rightarrow \quad c \in \mathbb{R} \text{ is a constant}$$

Thus $u(x,y) = -\frac{3}{2}y + c$ with $c \in \mathbb{R}$ and

$$f(x,y) = -\tfrac{3}{2}y + c + i\,\tfrac{3}{2}x$$

Q2 (b)

$$f(z) = \begin{cases} \dfrac{(\operatorname{Re} z)^4}{|z|^2} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

We need to show 2 things:

1) $\lim\limits_{z \to 0} f(z) = f(0) = 0$

2) $\lim\limits_{z \to z_0 \neq 0} f(z) = f(z_0) = \dfrac{(\operatorname{Re} z_0)^4}{|z_0|^2}$

let $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$. Then $|z| = r$, $\operatorname{Re} z = r\cos\theta$

$z \to 0$ means $r \to 0$, thus

1) $\lim\limits_{z \to 0} f(z) = \lim\limits_{r \to 0} \dfrac{(r\cos\theta)^4}{r^2} = \lim\limits_{r \to 0} \dfrac{r^4 \cos^4\theta}{r^2} = \lim\limits_{r \to 0} r^2 \cos^4\theta = 0$

Thus, f is continuous at $z = 0$

2) $\lim\limits_{z \to z_0 \neq 0} f(z) = \lim\limits_{(r,\theta) \to (r_0,\theta_0)} \dfrac{r^4 \cos^4\theta}{r^2} = \lim\limits_{(r,\theta) \to (r_0,\theta_0)} r^2 \cos^4\theta =$

$= r_0^2 \cos^4\theta_0 = f(z_0)$

Therefore f is continuous at every $z_0 \neq 0 \Rightarrow f$ is continuous for all $z \in \mathbb{C}$.

Q 3

(a) Root Test: Let $\sum_{n=0}^{\infty} a_n$, $a_n \in \mathbb{C}$. Consider the limit

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L.$$

- If L exists and $L<1$, then $\sum_{n=0}^{\infty} a_n$ converges absolutely
- If L exists and $L>1$, then $\sum_{n=0}^{\infty} a_n$ diverges
- If $L=1$ or the limit does not exist, then the test is inconclusive.

We have $$\sum_{n=2}^{\infty} \frac{i^n}{(\ln n)^n 72^n} (z+3)^n = \sum_{n=2}^{\infty} \left(\frac{i}{\ln n\, 72}\right)^n (z+3)^n$$

Let us apply the Root Test to find the radius of convergence

$$R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|a_n|}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{\left|\left(\frac{i}{\ln n\, 72}\right)^n\right|}} \underset{\downarrow}{=}$$

since for any $z_1, z_2 \in \mathbb{C}$ $|z_1 z_2| = |z_1||z_2|$

$$= \lim_{n\to\infty} \frac{1}{\sqrt[n]{\left|\frac{i}{\ln n\, 72}\right|^n}} = \lim_{n\to\infty} \frac{1}{\left|\frac{i}{\ln n\, 72}\right|} \underset{\downarrow}{=}$$

since if $z_2 \neq 0$ then $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

$$= \lim_{n\to\infty} \frac{1}{\frac{|i|}{|\ln n \cdot 72|}} \underset{\downarrow}{=} \lim_{n\to\infty} 72 \ln n = \infty$$

since $|i| = \sqrt{0^2+1^2} = 1$

Therefore the series converges for all $z \in \mathbb{C}$

Q3 (b) The branch points for $f(z)=\sqrt{z-z_0}$ are these for which $z-z_0=0$, namely $z=z_0$
The branch point of $\ln(z-z_0)$ is (in the same way) $z=z_0$
Thus we have the following branch points

$\ln(z^2-4)$: $z^2-4=0 \Leftrightarrow z^2=4 \Leftrightarrow z=\pm 2$
$\sqrt{z^2+5}$: $z^2+5=0 \Leftrightarrow z^2=-5$
let us find two solutions of $z^2=-5$:
let us write -5 in polar form
$|-5|=5$; $\cos\theta=\frac{-5}{5}=-1$, $\sin\theta=\frac{0}{5}=0 \Rightarrow \theta=\pi+2\pi k$

$\Rightarrow \sqrt{-5}=\sqrt{5}\left(\cos\left(\frac{\pi+2\pi k}{2}\right)+i\sin\left(\frac{\pi+2\pi k}{2}\right)\right)$ for $k=0,1$

$\Rightarrow k=0$ $\sqrt{5}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=i\sqrt{5}$
$k=1$ $\sqrt{5}\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)=-i\sqrt{5}$
Therefore, we have 4 branch points
$2, -2, i\sqrt{5}, -i\sqrt{5}$
Discontinuity points: we need to find the roots of the denominator
$$z-\sqrt{z^2+5}=0 \Leftrightarrow z=\sqrt{z^2+5} \Leftrightarrow z^2=z^2+5$$
From that $0=5$, namely there are no discontinuity points.

Q 4

(a) $f(z) = \dfrac{1}{(z-1)(z+8)}$

The laurent series of $f(z)$ about $z_0 = 1$ is given by

$$f(z) = \sum_{n=0}^{\infty} a_n (z-1)^n + \sum_{n=1}^{\infty} b_n (z-1)^{-n}$$

$$f(z) = \frac{1}{(z-1)(z+8)} = \frac{1}{9}\left(\frac{1}{z-1} - \frac{1}{z+8}\right) =$$

$$= \frac{1}{9}\left(\frac{1}{z-1} - \frac{1}{(z-1)+9}\right) = \frac{1}{9}\left(\frac{1}{z-1} - \frac{1}{9}\frac{1}{1+\left(\frac{z-1}{9}\right)}\right)$$

$$= \frac{1}{9}\left(\frac{1}{z-1} - \frac{1}{9}\underbrace{\frac{1}{1-\left(-\frac{z-1}{9}\right)}}_{\text{sum of the geometric series}}\right) =$$

$$= \frac{1}{9}\left(\frac{1}{z-1} - \frac{1}{9}\sum_{n=0}^{\infty}\left(-\frac{z-1}{9}\right)^n\right)$$

$$= \frac{1}{9}\left(\frac{1}{z-1} - \frac{1}{9}\underbrace{\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-1}{9}\right)^n}_{\text{valid on } 0<|z-1|<9}\right)$$

$$= \underbrace{\frac{1}{9}\cdot\frac{1}{z-1}}_{\text{the principal part}} - \frac{1}{81}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-1}{9}\right)^n$$

$\Rightarrow$ $b_1 = \frac{1}{9}$ and $b_n = 0$ for all $n > 1$

$a_n = -\frac{1}{81}\cdot\left(-\frac{1}{9}\right)^n \qquad n \geq 0$

Q4

b) <u>Prop</u>: $z = z_0$ is a pole of order $m > 0$ of a function $f \iff f$ can be expressed as $f(z) = \frac{\varphi(z)}{(z-z_0)^m}$, where

$\varphi(z)$ is analytic in a neighborhood of z_0 (and at z_0) and $\varphi(z_0) \neq 0$. Moreover

$$\text{Res}\,(f, z_0) = \frac{\varphi^{(m-1)}(z_0)}{(m-1)!}$$

In our case $f(z) = \frac{1}{(z-1)(z+8)}$. Rewrite

$f(z) = \frac{\varphi(z)}{z+8}$, where $\varphi(z) = \frac{1}{z-1}$. Then $\varphi(z)$ is

analytic in a neighborhood of $z=-8$ (and at $z=-8$) and $\varphi(-8) = -\frac{1}{9} \neq 0$. Thus $z=-8$ is a pole of order 1 of $f(z)$, namely simple pole. Thus

$m = 1$ $\quad \text{Res}\,(f; -8) = \frac{\varphi^{(0)}(-8)}{0!} = -\frac{1}{9}$

Q5

(a) $f(z) = \frac{1}{z} \cos \frac{1}{z^2}$

The only candidate for singularity is $z=0$

About $z=0$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$$

Thus

$$\cos \frac{1}{z^2} = 1 - \left(\frac{1}{z^2}\right)^2 \frac{1}{2!} + \left(\frac{1}{z^2}\right)^4 \frac{1}{4!} - \left(\frac{1}{z^2}\right)^6 \frac{1}{6!} + \dots$$

$$= 1 - \frac{1}{z^4}\frac{1}{2!} + \frac{1}{z^8}\frac{1}{4!} - \frac{1}{z^{12}}\frac{1}{6!} + \dots$$

Therefore, we obtain

$$\frac{1}{z}\cos\left(\frac{1}{z^2}\right) = \frac{1}{z} - \frac{1}{z^5}\frac{1}{2!} + \frac{1}{z^9}\frac{1}{4!} - \frac{1}{z^{13}}\frac{1}{6!} + \dots$$

Thus there is an isolated singularity at $z=0$ and since the principal part has infinitely many non-zero terms by definition it is an essential singularity.

Q5

(b) The Residue Theorem: Let f be an analytic function on and inside a simple, closed, positively oriented contour C, except at a finite number of isolated singularities $z_1, \ldots, z_n$ all inside C. Then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f, z_k)$$

We have from (a): $\text{Res}(f, 0) = 1$ (the coefficient of the term $\frac{1}{z}$)

f is analytic on and inside $|z| = 5$ (which is simple, closed, positively oriented contour) except at $z = 0$ that is inside the contour $|z| = 5$. Thus the Residue Theorem is applicable and we get

$$\int_{|z|=5} \frac{1}{z}\cos\frac{1}{z^2}\,dz \underset{\text{Residue Thm}}{=} 2\pi i \,\text{Res}(f, 0) = 2\pi i \cdot 1 = 2\pi i$$