Complex Variables Final exam 2021 Solutions

Q1

(a) $\frac{2z+3-4i}{i+1-4z} = -1+2i \iff 2z+3-4i = (-1+2i)(i+1-4z)$

$\iff 2z+3-4i = i-3+z(4-8i)$

$\iff z(2-8i) = 6-5i \iff z = \frac{6-5i}{2-8i}$

Let us multiply the numerator and the denominator by the complex conjugate of the denominator

$\overline{2-8i} = \overline{2} - \overline{8i} = 2+8i$

since for any $z_1, z_2 \in \mathbb{C}$ $\overline{z_1 - z_2} = \overline{z}_1 - \overline{z}_2$

Therefore

$$z = \frac{6-5i}{2-8i} \cdot \frac{2+8i}{2+8i} = \frac{26}{34} + i\,\frac{19}{34}$$

(b) $w = u+iv = \frac{1}{z} = \frac{1}{x+iy} \iff$

$$x+iy = \frac{1}{u+iv} = \frac{u}{u^2+v^2} - i\,\frac{v}{u^2+v^2}$$

Since for any $z_1, z_2 \in \mathbb{C}$ $z_1 = z_2 \iff \operatorname{Re} z_1 = \operatorname{Re} z_2$ and $\operatorname{Im} z_1 = \operatorname{Im} z_2$, we obtain

$$x = \frac{u}{u^2+v^2} \qquad y = -\frac{v}{u^2+v^2}$$

First, let us check what is the image of the boundaries: $x=1$, $x=4$

$x=4$: $\quad 4 = \frac{u}{u^2+v^2} \iff 4u^2+4v^2-u=0$

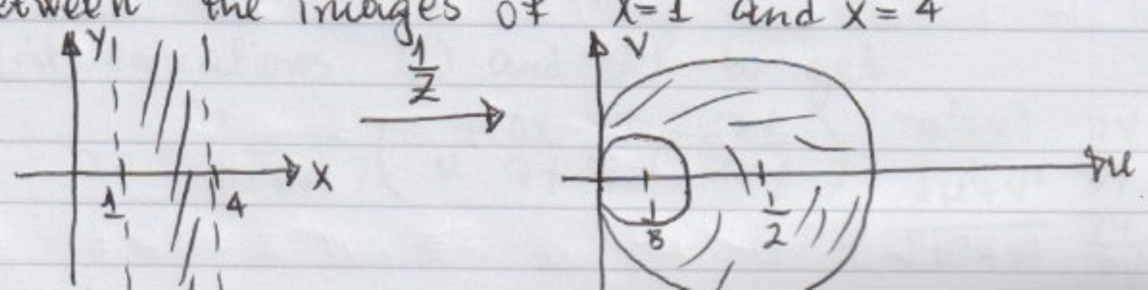
$\iff (u-\frac{1}{8})^2 + v^2 = \frac{1}{64}$

This is a circle of radius $\frac{1}{8}$ centered at $(\frac{1}{8}, 0)$

$x=1$: $\quad 1 = \frac{u}{u^2+v^2} \iff u^2+v^2-u=0 \iff (u-\frac{1}{2})^2+v^2 = \frac{1}{4}$

This is a circle of radius $\frac{1}{2}$ centered at $(\frac{1}{2}, 0)$

The image of the domain $1<x<4$ is the domain between the images of $x=1$ and $x=4$



Q 2

(a) f is an analytic function on Ω, thus CR equations hold for every $z \in \Omega$:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

By definition of the modulus

$$|f(z)| = \sqrt{u^2+v^2} \neq 0$$

Since $|f(z)|$ is a constant function we conclude that $\frac{\partial}{\partial x}|f(z)| = \frac{\partial}{\partial y}|f(z)| = 0$. Using the chain rule we obtain

$$0 = \frac{\partial}{\partial x}|f(z)| = \frac{\partial}{\partial x}\left(\sqrt{u^2(x,y)+v^2(x,y)}\right) = \qquad (1)$$

$$= \frac{2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}}{2\sqrt{u^2(x,y)+v^2(x,y)}} =$$

$$= \frac{1}{\sqrt{u^2+v^2}}\left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right)$$

In the same way we get

$$0 = \frac{\partial}{\partial y}|f(z)| = \frac{1}{\sqrt{u^2+v^2}}\left(u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y}\right) \qquad (2)$$

Using Cauchy-Riemann equations we obtain (from (1))

$$(3) \quad 0 = \frac{\partial}{\partial x}|f(z)| = \frac{1}{\sqrt{u^2+v^2}}\left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right) = \frac{1}{\sqrt{u^2+v^2}}\left(u\frac{\partial v}{\partial y} - v\frac{\partial u}{\partial y}\right)$$

let us multiply equation (2) by (v) and equation (3) by (u) and get

$$(4) \quad 0 = \frac{1}{\sqrt{u^2+v^2}}\left(uv\frac{\partial u}{\partial y} + v^2\frac{\partial v}{\partial y}\right)$$

$$(5) \quad 0 = \frac{1}{\sqrt{u^2+v^2}}\left(u^2\frac{\partial v}{\partial y} - uv\frac{\partial u}{\partial y}\right)$$

Add equations (4) and (5) to get

$$0 = \frac{1}{\sqrt{u^2+v^2}}\left(u^2\frac{\partial v}{\partial y} + v^2\frac{\partial v}{\partial y}\right) = \frac{u^2+v^2}{\sqrt{u^2+v^2}}\frac{\partial v}{\partial y}$$

$$= \sqrt{u^2+v^2}\,\frac{\partial v}{\partial y}$$

Since $|f(z)| = \sqrt{u^2+v^2} \neq 0$ we conclude that $\frac{\partial v}{\partial y} = 0$.

Using Cauchy-Riemann equations again we obtain (from (2))

$$(6) \quad 0 = \frac{\partial}{\partial y}|f(z)| = \frac{1}{\sqrt{u^2+v^2}}\left(u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y}\right) =$$

$$= \frac{1}{\sqrt{u^2+v^2}}\left(-u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}\right)$$

Multiply (1) by u and (6) by v and get

$$(7) \quad 0 = \left(\frac{\partial}{\partial y}|f(z)|\right)v = \frac{1}{\sqrt{u^2+v^2}}\left(vu\frac{\partial v}{\partial x} + u^2\frac{\partial u}{\partial x}\right)$$

$$(8) \quad 0 = \frac{1}{\sqrt{u^2+v^2}}\left(-uv\frac{\partial v}{\partial x} + v^2\frac{\partial u}{\partial x}\right)$$

Adding (7) and (8) we obtain

$$\sqrt{u^2+v^2}\,\frac{\partial u}{\partial x} = 0,$$

and since $\sqrt{u^2+v^2} \neq 0$ we conclude that $\frac{\partial u}{\partial x} = 0$.

If we multiply now equation (1) by v and equation (6) by $-u$ we get

$$(9) \quad \frac{1}{\sqrt{u^2+v^2}}\left(uv\frac{\partial u}{\partial x} + v^2\frac{\partial v}{\partial x}\right) = 0$$

$$(10) \quad \frac{1}{\sqrt{u^2+v^2}}\left(-uv\frac{\partial u}{\partial x} + u^2\frac{\partial v}{\partial x}\right) = 0$$

Adding (9) and (10) gives $\sqrt{u^2+v^2}\,\frac{\partial v}{\partial x} = 0 \Rightarrow \frac{\partial v}{\partial x} = 0$

Lastly, we multiply (2) by u and equation (3) by $-v$ we obtain

$$(11) \quad \frac{1}{\sqrt{u^2+v^2}}\left(u^2\frac{\partial u}{\partial y} + uv\frac{\partial v}{\partial y}\right) = 0$$

$$(12) \quad \frac{1}{\sqrt{u^2+v^2}}\left(-uv\frac{\partial v}{\partial y} + v^2\frac{\partial u}{\partial y}\right) = 0$$

Adding (11) and (12) gives: $\sqrt{u^2+v^2}\,\frac{\partial u}{\partial y} = 0 \Rightarrow \frac{\partial u}{\partial y} = 0$

Thus $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$, $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$, therefore u and v are constant functions, namely f is a constant function.

(b) The branch points of $f(z) = \sqrt{z - z_0}$ is $z = z_0$ (namely when $z - z_0 = 0$), thus we have 3 branch points

$$\sqrt{3z+2} : \quad 3z+2=0 \iff z = -\frac{2}{3}$$
$$\sqrt{z^2-4} : \quad z^2-4=0 \iff z=2 \text{ or } z=-2$$

The discontinuity points in this case are the roots of the denominator, namely z for which

$$z + \sqrt{z^2-4} = 0 \iff \sqrt{z^2-4} = -z \iff z^2-4 = z^2$$

Namely $-4=0$, which means that there are no discontinuity points.

Q 3

(a) Let $\sum_{n=0}^{\infty} a_n$, $a_n \in \mathbb{C}$. Consider the limit

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$$

- If the limit exists and $L < 1$, then $\sum_{n=0}^{\infty} a_n$ converges absolutely
- If the limit exists and $L > 1$, then $\sum_{n=0}^{\infty} a_n$ diverges
- If the limit does not exist or $L = 1$, then the test is inconclusive.

The radius of convergence R is given by

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

In that case $a_n = \frac{n^n}{n!}$, $a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$

Therefore

$$\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{n^n}{n!} : \frac{(n+1)^{n+1}}{(n+1)!}\right| =$$

$$= \lim_{n\to\infty} \left|\frac{n^n}{n!} \cdot \frac{(n+1)!}{(n+1)^{n+1}}\right| = \lim_{n\to\infty} \left| n^n \frac{n+1}{(n+1)^{n+1}}\right| =$$

$$= \lim_{n\to\infty} \left|\frac{n^n}{(n+1)^n}\right| = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n =$$

$$= \lim_{n\to\infty} \left(\frac{1}{\frac{n+1}{n}}\right)^n = \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^n =$$

$$\underset{\downarrow}{=} \frac{\lim_{n\to\infty} 1^n}{\lim_{n\to\infty} (1+\frac{1}{n})^n} \underset{\downarrow}{=} \frac{1}{e}$$

since $\lim_{n\to\infty} (1+\frac{1}{n})^n \neq 0$ $\quad \lim_{n\to\infty} (1+\frac{1}{n})^n = e$

Thus, the radius of convergence $R = \frac{1}{e}$, namely the series converges absolutely for $|z+3-4i| < \frac{1}{e}$

Q 3

(b) Let us plug $z = -\frac{11}{3} + i\frac{11}{3}$ and check whether the result is inside the disc of convergence:

$$\left|-\frac{11}{3} + i\frac{11}{3} + 3 - 4i\right| = \left|-\frac{2}{3} - i\frac{1}{3}\right| = \sqrt{\frac{4}{9}+\frac{1}{9}} = \frac{\sqrt{5}}{3}$$

Since $\sqrt{5} > 2$ we get $\frac{\sqrt{5}}{3} > \frac{2}{3} > \frac{1}{e}$, namely this point is outside of the disc of convergence, thus the series diverges.

Alternatively: for $z = -\frac{11}{3} + i\frac{11}{3}$ we get:

$$\sum_{n=0}^{\infty} (z+3-4i)^n \cdot \frac{n^n}{n!} = \sum_{n=0}^{\infty} \left(-\frac{2}{3} - i\frac{1}{3}\right)^n \frac{n^n}{n!}$$

Let us apply Ratio Test: $a_n = \frac{n^n}{n!}\left(-\frac{2}{3} - i\frac{1}{3}\right)^n$

Thus

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \underset{\substack{\uparrow \\ \text{in the same way as before}}}{=} \lim_{n\to\infty} \left|\left(\frac{n+1}{n}\right)^n \left(-\frac{2}{3} - i\frac{1}{3}\right)\right| =$$

$$= \left|-\frac{2}{3} - i\frac{1}{3}\right| \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \frac{\sqrt{5}}{3} \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e \cdot \frac{\sqrt{5}}{3}$$

Since the limit exists and $e\frac{\sqrt{5}}{3} > 1$ we conclude that the series diverges.

Q4

(a) About $z_0 = 0$ we have

$$\cos z = 1 - \frac{z^2}{2} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

Therefore, plugging $\frac{1}{z^2}$ instead of z we obtain

$$\cos\frac{1}{z^2} = 1 - \frac{1}{z^4}\cdot\frac{1}{2!} + \frac{1}{z^8}\frac{1}{4!} - \frac{1}{z^{12}}\frac{1}{6!} + \cdots$$

and

$$z^4\cos\frac{1}{z^2} = z^4\left(1 - \frac{1}{z^4}\frac{1}{2!} + \frac{1}{z^8}\frac{1}{4!} - \frac{1}{z^{12}}\frac{1}{6!} + \cdots\right)$$

$$= z^4 - \frac{1}{2!} + \frac{1}{z^4}\frac{1}{4!} - \frac{1}{z^8}\frac{1}{6!} + \frac{1}{z^{12}}\frac{1}{8!} - \cdots$$

$$= z^4 - \frac{1}{2!} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n+2)!}\frac{1}{z^{4n}}$$

Thus, we have

$a_0 = -\frac{1}{2}$, $a_1 = 0$, $a_2 = 0$, $a_3 = 0$, $a_4 = 1$, and for every $n \geq 5$ $a_n = 0$

$b_n = \frac{(-1)^{n-1}}{(2n+2)!}$ for $n = 4, 8, 12, 16, \dots$

$b_n = 0$ for other n.

To determine where this series converges we compute the radius of convergence:

$$R = \lim_{n\to\infty}\left|\frac{b_n}{b_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n-1}}{(2n+2)!}\cdot\frac{(2n+4)!}{(-1)^n}\right|$$

$$= \lim_{n\to\infty}(2n+4)(2n+3) = \infty,$$

therefore the series converges for all ~~z~~ $z \in \mathbb{C}$, in particular for $0 < |z| < 4$.

Q 4

(b) By definition the residue of a function $f(z)$ is the coefficient b_1 in the Laurent series expansion (corresponding to the term $\frac{1}{z-z_0}$)
Thus, since the coefficient of the term $\frac{1}{z}$ is equal to 0, we get

$$\operatorname{Res}\left(z^4 \cos \frac{1}{z^2}, 0\right) = 0$$

Q5

(a) The singularities are the singularities (or zeros) of the denominator and the singularities (if there are any) of the numerator.
Let us start with the denominator. We need to find all the solutions of $z^3 - i = 0 \Leftrightarrow z^3 = i$.

$|i| = 1$, $\arg i = \frac{\pi}{2}$, (since $\cos\theta = 0$, $\sin\theta = 1$)
Thus, we obtain

$$\sqrt[3]{i} = \sqrt[3]{1}\left(\cos\frac{\frac{\pi}{2}+2\pi k}{3} + i\sin\frac{\frac{\pi}{2}+2\pi k}{3}\right)$$

$$= \cos\left(\frac{\pi+4\pi k}{6}\right) + i\sin\left(\frac{\pi+4\pi k}{6}\right) \quad \text{for } k=0,1,2$$

These are isolated singularities.
There is an additional singularity (in the numerator) at $z=0$.
Let us determine the nature of these singularities:
$z=0$ is an isolated singularity. To determine its nature we look at the Laurent series expansion of e^{-1/z^3} about $z=0$ (Note: $z=0$ is not a root of the denominator and not a singularity of the denominator!)

$$e^{-1/z^2} = e^{-z^{-2}} = \sum_{n=0}^{\infty}\left(-\frac{1}{z^2}\right)^n\frac{1}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\cdot\frac{1}{z^{2n}}$$

$$= 1 - \frac{1}{z^2} + \frac{1}{2!\,z^4} - \frac{1}{3!\,z^6} + \cdots$$

Since the principal part consists of infinitely many non-zero terms by definition $z=0$ is an essential singularity.
Denote the 3 roots: $\sqrt[3]{i}$ by w_1, w_2, w_3.
Then:

(Now we study the roots of the denominator)
for $z = w_1$: Rewrite

$$f(z) = \frac{1}{z - w_1}\varphi(z), \text{ where } \varphi(z) = \frac{e^{-1/z^2}}{(z-w_2)(z-w_3)}$$

$\varphi(z)$ is analytic in a neighborhood of $z = w_1$, (and at $z = w_1$) { since $w_1 \neq w_2 \neq w_3$, $w_1 \neq w_3$ } and

$$\varphi(w_1) = \frac{e^{-1/w_1^2}}{(w_1 - w_2)(w_1 - w_3)} \neq 0.$$

(Note : by computation
$w_1 = \frac{\sqrt{3}}{2} + i\frac{1}{2}$, $w_2 = -\frac{\sqrt{3}}{2} + i\frac{1}{2}$, $w_3 = -i$)

Prop (from class) : $z = z_0$ is a pole of order $m > 0$ of a function f $\Longleftrightarrow$ f can be expressed as $f(z) = \frac{\varphi(z)}{(z-z_0)^m}$ where $\varphi(z)$ is analytic in a neighborhood of z_0 (and at z_0) and $\varphi(z_0) \neq 0$

Thus, by this proposition w_1 is a simple pole.

$z = w_2$: Rewrite $f(z) = \frac{\varphi(z)}{z - w_2}$, $\varphi(z) = \frac{e^{-1/z^2}}{(z-w_1)(z-w_3)}$

$\varphi(z)$ is analytic in a neighborhood of w_2 (and at w_2) and $\varphi(w_2) = \frac{e^{-1/w_2^2}}{(w_2 - w_1)(w_2 - w_3)} \neq 0$. Thus $z = w_2$ is a simple pole.

$z = w_3$: Rewrite $f(z) = \frac{\varphi(z)}{z - w_3}$, $\varphi(z) = \frac{e^{-1/z^2}}{(z-w_1)(z-w_2)}$

$\varphi(z)$ is analytic in a neighborhood of w_3 (and at w_3) and $\varphi(w_3) = \frac{e^{-1/w_3^2}}{(w_3 - w_1)(w_3 - w_2)} \neq 0$

Thus, by Proposition above $z = w_3$ is a simple pole.

Q5

(b) There are 2 singularities of $f(z) = \frac{\cos z}{(z-1)(z-2)^2}$

$z=1, z=2$. Since our contour is a circle of radius $\frac{3}{2}$ around 3 $z=1$ is outside of that circle, therefore does not contribute to the value of the integral:

Residue Theorem: let f be analytic on and inside a simple, closed, positively oriented contour C, except at a finite number of isolated singularities $z_1, .., z_n$ all inside C. Then

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}\ (f; z_k)$$

Let us determine the nature of $z=2$:

Rewrite $f(z) = \frac{\varphi(z)}{(z-2)^2}$, where $\varphi(z) = \frac{\cos z}{z-1}$.

Since $\varphi(z)$ is analytic in a neighborhood of $z=2$ (and at $z=2$) and since $\varphi(2) = \frac{\cos 2}{1} \neq 0$, by Proposition formulated in Q4 (b) $z=2$ is a pole of order 2.

If z_0 is a pole of order m, we proved that

$$\text{Res}\ (f; z_0) = \frac{\varphi^{(m-1)}(z_0)}{(m-1)!}$$

Thus, we obtain for $m=2$

$$\text{Res}\ (f; 2) = \frac{\varphi'(2)}{1!} = \frac{-\sin 1(2-1) - \cos 1}{1!\ (2-1)^2} = -\sin 1 - \cos 1$$

$$\left(\frac{\cos z}{(z-1)}\right)' = \frac{-\sin z(z-1) - \cos z}{(z-1)^2}$$

Therefore

$$\int_{|z-3|=\frac{3}{2}} \frac{\cos z}{(z-1)(z-2)^2} dz = -2\pi i (\sin 1 + \cos 1)$$

by the Residue theorem