Complex Variables Final Exam 2020 Solutions

(Q1) a) Since $i+i^2+i^3+i^4 = i-1-i+1=0$, we get

$$i+i^2+\cdots+i^6 = i-1$$

Since for any $z_1, z_2 \in \mathbb{C}$ $\quad |z_1 z_2| = |z_1||z_2|$ and given $z_2 \neq 0$ $\quad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ we obtain

$$\left|\frac{(2-2i)^2(3i-4)}{(i+i^2+\cdots+i^6)^6}\right| = \frac{|2-2i|^2|3i-4|}{|i-1|^6} = \frac{\left(\sqrt{2^2+(-2)^2}\right)^2\sqrt{3^2+(-4)^2}}{\left(\sqrt{1^2+(-1)^2}\right)^6}$$

$$= \frac{8\cdot 5}{2^{\frac{1}{2}\cdot 6}} = \frac{8\cdot 5}{2^3} = \frac{8\cdot 5}{8} = 5$$

(b) $z^7 = 4+3i \iff z = \sqrt[7]{4+3i}$. Let us write $4+3i$ in polar form:

$|4+3i| = \sqrt{4^2+3^2} = 5 \qquad \cos\theta = \frac{4}{5}, \ \sin\theta = \frac{3}{5} \Rightarrow$

$\theta = \arccos\frac{4}{5} + 2\pi k$ (or $\arcsin\frac{3}{5} + 2\pi k$)

Thus

$$\sqrt[7]{4+3i} = 5^{1/7}\left(\cos\left(\frac{\arccos\left(\frac{4}{5}\right)+2\pi k}{7}\right) + i\sin\left(\frac{\arccos\left(\frac{4}{5}\right)+2\pi k}{7}\right)\right)$$

for $k = 0,1,2,3,4,5,6$.

(Q2)(a) Let $\sum_{n=0}^{\infty} a_n$, $a_n \in \mathbb{C}$. Consider the limit

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = L$$

- If the limit exists and $L<1$, then $\sum_{n=0}^{\infty} a_n$ converges absolutely
- If the limit exists and $L>1$, then $\sum_{n=0}^{\infty} a_n$ diverges
- If the limit does not exist or $L=1$, the test is inconclusi

(b) The Radius of convergence

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n!/n^n}{(n+1)!/(n+1)^{n+1}}\right| = \lim_{n\to\infty}\frac{n!}{n^n}\cdot\frac{(n+1)^{n+1}}{(n+1)!}$$

$$= \lim_{n\to\infty}\frac{1}{n^n}\frac{(n+1)^{n+1}}{n+1} = \lim_{n\to\infty}\frac{(n+1)^n}{n^n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

Thus, $R=e$, namely the series converges for

$$|z-2i|<e$$

(c) Let us plug $z=2i-2$ and check whether the result is inside the disc of convergence:

$$|2i-2-2i| = |-2| = 2 < e,$$

thus the series converges absolutely for $z=2i-2$

(Q3) (a) We use the Taylor series expansion for $\sin z$ about $z=0$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots$$

Plugging $\frac{1}{z}$ instead of z gives us

$$\sin\frac{1}{z} = \frac{1}{z} - \frac{1}{3!}\frac{1}{z^3} + \frac{1}{5!}\frac{1}{z^5} - \frac{1}{7!}\frac{1}{z^7} + \dots$$

Thus

$$z^2 \sin\frac{1}{z} = z^2\left(\frac{1}{z} - \frac{1}{3!}\frac{1}{z^3} + \frac{1}{5!}\frac{1}{z^5} - \frac{1}{7!}\frac{1}{z^7} + \dots\right)$$

$$= z - \frac{1}{3!}\frac{1}{z} + \frac{1}{5!}\frac{1}{z^3} - \frac{1}{7!}\frac{1}{z^5} + \dots$$

By definition the residue is the coefficient of the term $\frac{1}{z} = \frac{1}{z-0}$ in the expansion, therefore

$$\operatorname{Res}\left(z^2 \sin\frac{1}{z}, 0\right) = -\frac{1}{3!} = -\frac{1}{6}.$$

(b) From (a) we get

$$z^2 \sin\frac{1}{z} = z - \frac{1}{3!}\frac{1}{z} + \frac{1}{5!}\frac{1}{z^3} - \frac{1}{7!}\frac{1}{z^5} + \dots$$

$$= z + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!}\frac{1}{z^{2n-1}},$$

therefore $a_0 = 0,\ a_1 = 1,\ a_n = 0$ for all $n > 1$

$$b_n = \frac{(-1)^n}{(2n+1)!} \qquad n \geq 1$$

The radius of convergence

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n}{(2n+1)!}\cdot\frac{(2n+3)!}{(-1)^{n+1}}\right| =$$

$$= \lim_{n\to\infty}(2n+3)(2n+2) = \infty$$

Thus, the series converges for all $z \neq 0 \in \mathbb{C}$.

([illegible]) (a) Let $z = x+iy$, $f(z) = f(x,y) = u(x,y)+iv(x,y)$
Then $\operatorname{Im} z = y$, $\operatorname{Re} f(z) = u(x,y)$, namely we have $u(x,y) = y$. We are looking for all the analytic functions of the form

$$f(x,y) = y + iv(x,y).$$

Since $f(x,y)$ is analytic, u and v are differentiable and CR equations hold

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} = 0 = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = 1 = -\frac{\partial v}{\partial x}$$

Integration with respect to x of $\frac{\partial v}{\partial x}$ gives

$$v(x,y) = \int \frac{\partial v}{\partial x}\, dx = \int -1\, dx = -x + c(y)$$

To find $c(y)$ we differentiate $v(x,y)$ with respect to y and compare to $\frac{\partial v}{\partial y}$ obtained from CR equations

$$\frac{\partial v}{\partial y} = c'(y) = 0 \quad \Rightarrow \quad c \in \mathbb{R} \text{ is a constant.}$$

Thus $v(x,y) = -x + c$ with $c \in \mathbb{R}$ and

$$f(x,y) = y + i(-x+c)$$

(b) The singularities are the zeros of the denominator (and the singularities of the denominator) and the singularities (if there are any) of the numerator.
The denominator

$$z^2 - \frac{\pi^2}{4} = 0 \iff \left(z - \frac{\pi}{2}\right)\left(z + \frac{\pi}{2}\right) = 0 \iff z = \frac{\pi}{2} \text{ or } z = -\frac{\pi}{2}$$

These are 2 isolated singularities.
There is an additional singularity (in the numerator) at $z = 0$.
Let us determine the nature of these singularities:

$z=0$ is an isolated singularity. To determine its nature we look at the Laurent series expansion about $z=0$:

$$e^{-\frac{1}{z}} = e^{-z^{-1}} = \sum_{n=0}^{\infty} \left(-\frac{1}{z}\right)^n \frac{1}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{1}{z^n} =$$

$$= 1 - \frac{1}{z} + \frac{1}{2!z^2} - \frac{1}{3!\,z^3} + \frac{1}{4!\,z^4} - \dots$$

Since the principal part consists of infinitely many non-zero terms by definition it is an <u>essential</u> singularity.

$z=\frac{\pi}{2}$: Rewrite: $f(z) = \frac{1}{z-\frac{\pi}{2}} \varphi(z)$, where $\varphi(z) = \frac{e^{-1/z}}{z+\frac{\pi}{2}}$

$\varphi(z)$ is analytic in a neighborhood of $z=\frac{\pi}{2}$ (and at $z=\frac{\pi}{2}$), and $\varphi(\frac{\pi}{2}) = \frac{e^{-2/\pi}}{\frac{\pi}{2}+\frac{\pi}{2}} \neq 0$ $\left(= \frac{1}{\pi} e^{-\frac{2}{\pi}}\right)$,

therefore by Prop ⊛ (below) $z=\frac{\pi}{2}$ is a simple pole.

$z=-\frac{\pi}{2}$: Rewrite $f(z) = \frac{1}{z+\frac{\pi}{2}} \varphi(z)$, where $\varphi(z) = \frac{e^{-1/z}}{z-\frac{\pi}{2}}$.

$\varphi(z)$ is analytic in a neighborhood of $z=-\frac{\pi}{2}$ (and at $z=-\frac{\pi}{2}$) and $\varphi(-\frac{\pi}{2}) = -\frac{e^{2/\pi}}{\pi} \neq 0$. Thus $z=-\frac{\pi}{2}$ is a simple pole (by Prop ⊛ below).

Prop ⊛: $z=z_0$ is a pole of order $m>0$ of $f \iff f$ can be expressed as $f(z) = \frac{\varphi(z)}{(z-z_0)^m}$ where $\varphi(z)$ is analytic at a neighborhood of z_0 (and at z_0) and $\varphi(z_0) \neq 0$.

(Q5) (a) Let f be analytic on and inside a simple, closed, positively oriented contour C, except at a finite number of isolated singularities $z_1, \ldots, z_n$ all inside C. Then $\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f; z_k)$.

(b) There are 2 singularities $z=0$, $z=1$.
Let us first determine the nature of these singularities and compute $\operatorname{Res}(f;0)$, $\operatorname{Res}(f;1)$ ($z=0$ and $z=1$ both inside $|z|=2$ (our contour), thus both contribute to the value of the integral.

$z=0$: Let us look at the expansion of $\sin z$ about $z=0$:

$$\frac{\sin z}{z(z-1)^2} = \frac{1}{z(z-1)^2}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots\right) =$$
$$= \frac{1}{(z-1)^2}\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \ldots\right) \equiv \varphi(z)$$

$\varphi(0) = 1 \neq 0 \Rightarrow z=0$ is a removable singularity and $\operatorname{Res}(f;0)=0$

$z=1$: Rewrite $f(z) = \frac{1}{(z-1)^2}\varphi(z)$, where $\varphi(z) = \frac{\sin z}{z}$
$\varphi(z)$ is analytic in a neighborhood of $z=1$ (and at $z=1$) and $\varphi(1) = \sin 1 \neq 0$, therefore by Prop (*) above $z=1$ is a pole of order 2. In the case z_0 is a pole of order m: $\operatorname{Res}(f, z_0) = \frac{\varphi^{(m-1)}(z_0)}{(m-1)!}$

Thus, we obtain here for $m=2$

$$\operatorname{Res}(f,1) = \frac{\varphi'(1)}{1!} = \frac{1\cdot\cos 1 - \sin 1}{1} = \cos 1 - \sin 1$$

residue theorem