1

Complex Variables 2019 Sample exam Solutions

Q1

a) $(z-4i)(3i-5)=(2i-z)(1+i)$

$\Rightarrow \quad 3iz-12i^2-5z+20i = 2i-z+2i^2-zi$

since $i^2=-1$ we obtain

$$3iz+12-5z+20i = 2i-z-2-zi$$

$\Rightarrow \quad 3iz-5z+z+zi = 2i-2-12-20i$

$\Rightarrow \quad 4iz-4z = -14-18i$

$\rightarrow$ (Divide both sides by 2): $2iz-2z=-7-9i$

Namely,

$$z(2i-2) = -(7+9i)$$

$$z = -\frac{7+9i}{2i-2}$$

Now we multiply and divide by $\overline{2i-2} = \overline{2i}-\overline{2} = -2i-2$:

$$z = -\frac{7+9i}{2i-2}\cdot\frac{-2i-2}{-2i-2} = -\frac{-14i-18i^2-14-18i}{-4i^2+4i-4i+4} =$$

$$= -\frac{-14i+18-14-18i}{4+4} = -\frac{4-32i}{8} = -\frac{1}{2}+4i$$

let us plug $z=-\frac{1}{2}+4i$ back into the initial equation:

$$\left(-\frac{1}{2}+4i-4i\right)(3i-5) = \left(2i+\frac{1}{2}-4i\right)(1+i)$$

$$-\frac{3}{2}i+\frac{5}{2} = \left(\frac{1}{2}-2i\right)(1+i)$$

$$-\frac{3}{2}i+\frac{5}{2} = \frac{1}{2}-2i+\frac{1}{2}i-2i^2 = \frac{1}{2}-\frac{3}{2}i+2 = -\frac{3}{2}i+\frac{5}{2}$$

Q1

b) $|z| = \operatorname{Re} z + \frac{1}{4}$

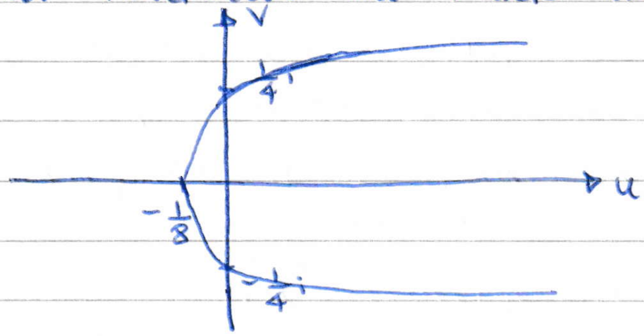
Since $|z| = \sqrt{x^2+y^2}$ for $z = x+iy$ and $\operatorname{Re} z = x$, we get:

$$\sqrt{x^2+y^2} = x + \frac{1}{4} \iff x^2+y^2 = \left(x+\frac{1}{4}\right)^2 = x^2 + \frac{1}{2}x + \frac{1}{16}$$

$$\iff y^2 = \frac{1}{2}x + \frac{1}{16} \iff \frac{1}{2}x = y^2 - \frac{1}{16} \iff x = 2y^2 - \frac{1}{8}$$

(or $y^2 = \frac{1}{2}x + \frac{1}{16}$)

This is rightward opening parabola with vertex $\left(-\frac{1}{8}, 0\right)$, that intersects the v-axis at $\left(0, \pm\frac{1}{4}\right)$



③

Q2

a) let $\sum_{n=0}^{\infty} a_n$, $a_n \in \mathbb{C}$, Consider the limit
$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$$

- If L exists and $L<1$, then $\sum_{n=0}^{\infty} a_n$ converges absolutely
- If L exists and $L>1$, then $\sum_{n=0}^{\infty} a_n$ diverges
- If L does not exist or $L=1$, then the test is inconclusive

b) let us apply Cauchy-Hadamard formula:

$$R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|a_n|}} \underset{a_n=(1+i)^n}{=} \lim_{n\to\infty} \frac{1}{\sqrt[n]{|(1+i)^n|}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|1+i|^n}} =$$

$$= \lim_{n\to\infty} \frac{1}{|1+i|} = \frac{1}{|1+i|} \underset{|1+i|=\sqrt{1^2+1^2}=\sqrt{2}}{=} \frac{1}{\sqrt{2}}$$

Thus, the radius of convergence is $\frac{1}{\sqrt{2}}$, namely for $|z|<\frac{1}{\sqrt{2}}$ the series converges absolutely.

c) For any z with $|z|=\frac{1}{\sqrt{2}}$ (on the boundary of the disc of convergence) the sequence $\{(1+i)^n z^n\}_{n=0}^{\infty}$ does not converge to 0, therefore the series diverges by the Test for divergence.

If $|z|=\frac{1}{\sqrt{2}}$: let us represent z in polar coordinates:
$$z = \frac{1}{\sqrt{2}}(\cos\theta + i\sin\theta) \Rightarrow z^n = 2^{-\frac{n}{2}}(\cos n\theta + i\sin(n\theta))$$

Thus, for any z with $|z|=\frac{1}{\sqrt{2}}$

$$|(1+i)^n z^n| = |(1+i)^n 2^{-\frac{n}{2}}(\cos n\theta + i\sin n\theta)| \underset{(1+i)^n = 2^{n/2}(\cos\frac{\pi}{4}n + i\sin\frac{\pi}{4}n)}{=}$$

$$= |2^{\frac{n}{2}} 2^{-\frac{n}{2}}(\cos n\theta + i\sin n\theta)(\cos\tfrac{\pi}{4}n + i\sin\tfrac{\pi}{4}n)|$$

$$= |\cos(n(\theta+\tfrac{\pi}{4})) + i\sin(n(\theta+\tfrac{\pi}{4}))| = \sqrt{\cos^2(n(\theta+\tfrac{\pi}{4})) + \sin^2(n(\theta+\tfrac{\pi}{4}))}$$

$$= 1 \underset{n\to\infty}{\not\to} 0$$

Q3

a) First of all, note that

$$f(z) = \frac{1}{(z-4)(z+2)} = \frac{1}{6}\left(\frac{1}{z-4} - \frac{1}{z+2}\right)$$

laurent series of $f(z)$ about $z_0 = 4$ is given by

$$\sum_{n=0}^{\infty} a_n (z-4)^n + \sum_{n=1}^{\infty} b_n (z-4)^{-n}$$

$$f(z) = \frac{1}{6}\left(\frac{1}{z-4} - \frac{1}{z+2}\right) = \frac{1}{6}\left(\frac{1}{z-4} - \frac{1}{6+(z-4)}\right)$$

$$= \frac{1}{6}\left(\frac{1}{z-4} - \frac{1}{6}\,\frac{1}{1-\left(-\frac{z-4}{6}\right)}\right)$$

On a punctured disc $0 < |z-4| < 6$ we have

$$\frac{1}{1-\left(-\frac{z-4}{6}\right)} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-4}{6}\right)^n$$: this series is valid if

$\left|\frac{z-4}{6}\right| < 1$, namely exactly for $0 < |z-4| < 6$. Therefore:

$$f(z) = \frac{1}{6}\left(\frac{1}{z-4} - \frac{1}{6}\sum_{n=0}^{\infty} (-1)^n \left(\frac{z-4}{6}\right)^n\right) = -\sum_{n=0}^{\infty} (-1)^n \frac{(z-4)^n}{6^{n+2}}$$

$$+ \underbrace{\frac{1/6}{z-4}}_{\text{the priciple part of the laurent series}}$$

b) From (a): $b_1 = \frac{1}{6}$, thus $\mathrm{Res}(f, 4) = \frac{1}{6}$.

Alternatively:

$z = 4$ is a simple pole, since $f(z) = \frac{1}{z-4}\varphi(z)$, $\varphi(z) = \frac{1}{z+2}$ is analytic at a neighborhood of $z_0 = 4$ (and at $z_0 = 4$) and $\varphi(4) = \frac{1}{6} \neq 0$. Thus, by what we proved

$$\mathrm{Res}(f, 4) = \frac{\varphi^{(m-1)}(4)}{(m-1)!} \underset{m=1}{=} \frac{\varphi(4)}{0!} = \frac{1}{6}.$$

Q4

a) $u(x,y) = 4e^x \sin y$

$$\frac{\partial u}{\partial x} = 4e^x \sin y \qquad \frac{\partial u}{\partial y} = 4e^x \cos y$$

$$\frac{\partial^2 u}{\partial x^2} = 4e^x \sin y \qquad \frac{\partial^2 u}{\partial y^2} = -4e^x \sin y$$

$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4e^x \sin y - 4e^x \sin y = 0$, thus $u(x,y)$ is harmonic.

We find its harmonic conjugate by using the CR equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ we obtain

$$v(x,y) = \int \frac{\partial v}{\partial y}\,dy = \int 4e^x \sin y\,dy = -4e^x \cos y + c(x)$$

To find $c(x)$: differentiate $v(x,y)$ with respect to x and compare the result with $\frac{\partial v}{\partial x}$ obtained from the CR equations:

$$\frac{\partial v}{\partial x} = -4e^x \cos y + c'(x) = -\frac{\partial u}{\partial y} = -4e^x \cos y$$

Thus, c is a real constant and we get $v(x,y) = -4e^x \cos y + c$.

⑥

Q4.b) $f(z) = \dfrac{e^{-iz}}{z^2 - \frac{\pi^2}{9}} = \dfrac{e^{-iz}}{(z-\frac{\pi}{3})(z+\frac{\pi}{3})}$

Therefore, $f(z)$ has 2 isolated singularities $z = \pm\frac{\pi}{3}$.

$z = \frac{\pi}{3}$: Rewrite $f(z) = \dfrac{1}{z-\frac{\pi}{3}}\varphi(z)$, where $\varphi(z) = \dfrac{e^{-iz}}{z+\frac{\pi}{3}}$

analytic at a neighborhood of $z = \frac{\pi}{3}$ (and at $z = \frac{\pi}{3}$)

and

$$\varphi\left(\frac{\pi}{3}\right) = \frac{e^{-i\frac{\pi}{3}}}{\frac{2\pi}{3}} = \frac{3}{2\pi}\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$$

$$= \frac{3}{2\pi}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) \neq 0$$

Thus, by Proposition ⊛ $z = \frac{\pi}{3}$ is a simple pole.

$z = -\frac{\pi}{3}$: Rewrite $f(z) = \dfrac{1}{z+\frac{\pi}{3}}\varphi(z)$, where $\varphi(z) = \dfrac{e^{-iz}}{z-\frac{\pi}{3}}$ -

analytic at a neighborhood of $z = -\frac{\pi}{3}$ (and at $z = -\frac{\pi}{3}$)

and

$$\varphi\left(-\frac{\pi}{3}\right) = \frac{e^{i\frac{\pi}{3}}}{-\frac{2\pi}{3}} = -\frac{3}{2\pi}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) =$$

$$= -\frac{3}{2\pi}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \neq 0$$

Thus, by Proposition ⊛ $z = -\frac{\pi}{3}$ is a simple pole.

Proposition ⊛: $z = z_0$ is a pole of order $m > 0$ of $f \Leftrightarrow$ f can be expressed as $f(z) = \dfrac{\varphi(z)}{(z-z_0)^m}$ where $\varphi(z)$ is analytic at a neighborhood of z_0 (and at z_0) and $\varphi(z_0) \neq 0$.

Moreover, $\operatorname{Res}(f, z_0) = \dfrac{\varphi^{(m-1)}(z_0)}{(m-1)!}$

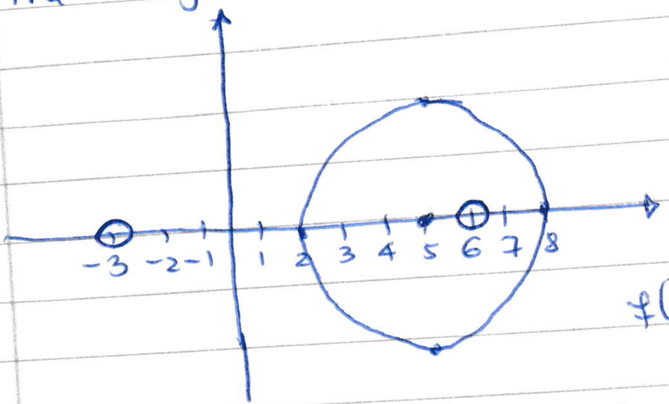
Q5

a) Residue Theorem: let f be analytic on and inside a simple, closed, positively oriented contour C, except at a finite number of isolated singularities $z_1, z_2, .., z_n$ all inside C.

Then, $\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f, z_k)$

b) $\int_{|z-5|=3} \frac{z-1}{(z+3)(z-6)^3} dz$

The singularities are at $z=-3$ and at $z=6$.



$z=-3$ is outside of $|z-5|=3$, thus does not contribute to the value of this integral.

$z=6$ is a pole of order 3, since

$f(z) = \frac{z-1}{(z+3)(z-6)^3} = \frac{\varphi(z)}{(z-6)^3}$,

where $\varphi(z) = \frac{z-1}{z+3}$ is analytic at a neighborhood of $z=6$ (and at $z=6$) and $\varphi(6) = \frac{5}{9} \neq 0$

By Proposition ⊛ (formulated in Q4)

$\underset{\downarrow}{\text{Res}(f,6)} = \frac{\varphi''(6)}{2!}$

$m=3$

$\varphi'(z) = \frac{z+3-(z-1)}{(z+3)^2} = \frac{4}{(z+3)^2}$ $\quad \varphi''(z) = -\frac{8}{(z+3)^3}$

$\Rightarrow \varphi''(6) = -\frac{8}{9^3} = -\frac{8}{729}$ $\Rightarrow \text{Res}(f;6) = \frac{-\frac{8}{729}}{2!} = -\frac{4}{729}$

Thus, by Residue Theorem

$\int_{|z-5|=3} \frac{z-1}{(z+3)(z-6)^3} dz = 2\pi i \, \text{Res}(f;6) = 2\pi i \left(-\frac{4}{729}\right)$

$= -\frac{8\pi}{729} i$

Alternatively: ⊛ By extended Cauchy Integral formula for $n=2$:

$$\varphi''(6) = \frac{2!}{2\pi i} \int_{|z-5|=3} \frac{\varphi(z)}{(z-6)^3}\, dz \qquad \varphi(z) = \frac{z-1}{z+3}$$

$$\Rightarrow \int_{|z-5|=3} \frac{\varphi(z)}{(z-6)^3}\, dz = \frac{2\pi i}{2!}\, \varphi''(6) = \pi i \left(-\frac{8}{729}\right) = -\frac{8\pi i}{729}$$

(*) $\varphi(z)$ is analytic on and everywhere inside $|z-5|=3$; $|z-5|=3$ is simple, closed, positively oriented contour, $z=6$ is inside C