Complex Variables Late Summer (2018) Solution

(Q1) a) Compute $\left| \dfrac{(3+4i)(1+i)^6}{i^5(2+4i)^2} \right|$.

First, by De Moivre formula for $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$ we have $z^n = r^n(\cos n\theta + i\sin n\theta)$.

let us compute $(1+i)^6$:

$|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$ $\quad \cos\theta = \frac{1}{\sqrt{2}}$, $\sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}\ (+2\pi k)$

Therefore,

$$1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \Rightarrow (1+i)^6 = (\sqrt{2})^6\left(\cos\frac{6\pi}{4} + i\sin\frac{6\pi}{4}\right) = 8\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$$
$$= 8(0 + i(-1)) = -8i$$

Next: $i^5 = i$ and $(2+4i)^2 = -12+16i = 4(-3+4i) = -4(3-4i)$

Therefore, we get

$$\left|\frac{(3+4i)(1+i)^6}{i^5(2+4i)^2}\right| = \left|\frac{-8i(3+4i)}{-4i(3-4i)}\right| = \frac{|-8i||3+4i|}{|-4i||3-4i|} = \frac{8}{4}\,\frac{\sqrt{9+16}}{\sqrt{9+16}} = 2.$$

b) $z^5 = 32$, namely $z = \sqrt[5]{32}$. Write 32 in polar form

$|32| = 32$ $\quad \arg 32 = 0\ (+2\pi k)$ since $\cos\theta = \frac{32}{32} = 1$, $\sin\theta = \frac{0}{32} = 0$.

Therefore: $\quad 32 = 32(\cos(2\pi k) + i\sin(2\pi k))$.

Thus,

$$\sqrt[5]{32} = 32^{1/5}\left(\cos\frac{0+2\pi k}{5} + i\sin\frac{0+2\pi k}{5}\right) = 2\left(\cos\frac{2\pi k}{5} + i\sin\frac{2\pi k}{5}\right)$$

for $k = 0, 1, 2, 3, 4$

$k=0 \quad 2\left(\cos\frac{0}{5} + i\sin\frac{0}{5}\right) = 2$

$k=1 \quad 2\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)$

$k=2 \quad 2\left(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right)$

$k=3 \quad 2\left(\cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}\right)$

$k=4 \quad 2\left(\cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}\right)$

(Q 2) a) Let $z = x+iy$. Then $f(z) = |z|^4 = (x^2+y^2)^2$

Namely, if we denote $f(z) = u(x,y)+iv(x,y)$, then $u(x,y) = (x^2+y^2)^2$ and $v(x,y)=0$. Let us check where u and v are differentiable and Cauchy-Riemann equations hold. We have

$$\frac{\partial u}{\partial x} = 4x(x^2+y^2) \qquad \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial y} = 4y(x^2+y^2) \qquad \frac{\partial v}{\partial x} = 0$$

$$\Rightarrow \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \Leftrightarrow \begin{cases} 4x(x^2+y^2)=0 \\ 4y(x^2+y^2)=0 \end{cases} \Leftrightarrow \begin{matrix}(1)\\(2)\end{matrix}\begin{cases} x(x^2+y^2)=0 \\ y(x^2+y^2)=0 \end{cases}$$

If $x=0$, then we obtain from (2): $y^3=0 \Leftrightarrow y=0$

If $y=0$, then we obtain from (1): $x^3=0 \Leftrightarrow x=0$

Note that since $x^2, y^2 \geq 0$ for any $x,y \in \mathbb{R}$ we get $x^2+y^2=0 \Leftrightarrow x=y=0$.

Therefore, Cauchy-Riemann equations hold only at $z=0$, thus $f(z)=|z|^4$ is not differentiable at $z \neq 0$. $f(z)$ is differentiable at $z=0$ since $u(x,y)$ and $v(x,y)$ are defined in a neighborhood of $(0,0)$, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are defined in a neighborhood of $(0,0)$ (there exists such neighborhood) and continuous at $(0,0)$ and the Cauchy-Riemann equations hold.

b) We have $\operatorname{Re} f(z) = x^2-y^2+2x$ and $f(i) = 2i-1$

Since f is analytic Cauchy-Riemann equations hold. Namely, if we denote $f(z) = u(x,y)+iv(x,y)$, we are looking for $v(x,y)$ s.t.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

We have

$$\frac{\partial u}{\partial x} = 2x+2 = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

Let us integrate $\frac{\partial v}{\partial y}$ with respect to y (to reconstruct $v(x,y)$):

$$v(x,y) = \int \frac{\partial v}{\partial y}\,dy = \int (2x+2)\,dy = 2xy+2y+c(x).$$

Therefore: $\frac{\partial v}{\partial x} = 2y + c'(x)$ and $-\frac{\partial v}{\partial x} = -2y - c'(x)$

On the other hand $\frac{\partial v}{\partial x} = -2y$, namely $-2y = -2y - c'(x)$, namely $c'(x)=0$, namely c is a constant. We get:

$f(x,y) = x^2 - y^2 + 2x + i(2xy + 2y + c)$

We find the value of c using $f(i)$:

$$f(i) = 0 - 1^2 + 2\cdot 0 + i(2\cdot 0\cdot 1 + 2\cdot 1 + c) = -1 + i(2+c) \underset{\text{given}}{=} 2i - 1$$

Therefore $c = 0$ and

$$f(x,y) = x^2 - y^2 + 2x + i(2xy + 2y) = x^2 - y^2 + i2xy + 2(x+iy) = z^2 + 2z.$$

(Q3) a) let $\sum_{n=0}^{\infty} a_n$, $a_n \in \mathbb{C}$. Consider the following limit $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$

- If L exists and $L < 1$, then $\sum_{n=0}^{\infty} a_n$ converges absolutely
- If L exists and $L > 1$, then $\sum_{n=0}^{\infty} a_n$ diverges
- If L does not exist or $L = 1$, then the test is inconclusive.

b) $\sum_{n=1}^{\infty} \left(\frac{z}{in}\right)^n$: We apply the Root Test as follows: $a_n = \left(\frac{z}{in}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{z}{in}\right|^n} = \lim_{n\to\infty} \frac{|z|}{|in|} = |z| \lim_{n\to\infty} \frac{1}{n} = |z| \cdot 0 = 0 < 1$$ for any z! Thus, this series converges for any $z \in \mathbb{C}$ ($R = \infty$).

(Q4) a) First of all, note that

$$f(z) = \frac{1}{(z-6)(z+3)} = \frac{1}{9}\left(\frac{1}{z-6} - \frac{1}{z+3}\right)$$

Laurent series of $f(z)$ about $z_0 = -3$ is given by

$$\sum_{n=0}^{\infty} a_n (z+3)^n + \sum_{n=1}^{\infty} b_n (z+3)^{-n}.$$

$$f(z) = \frac{1}{9}\left(\frac{1}{z-6} - \frac{1}{z+3}\right) = \frac{1}{9}\left(\frac{1}{-9+(z+3)} - \frac{1}{z+3}\right) = \frac{1}{9}\left(-\frac{1}{9}\frac{1}{1-\frac{z+3}{9}} - \frac{1}{z+3}\right)$$

On a punctured disc $0 < |z+3| < 9$ we have

$$\frac{1}{1 - \frac{z+3}{9}} = \sum_{n=0}^{\infty} \left(\frac{z+3}{9}\right)^n$$: this series is valid if $\left|\frac{z+3}{9}\right| < 1$,

namely, exactly for $0 < |z+3| < 9$. Therefore

$$f(z) = \frac{1}{9}\left(-\frac{1}{9}\sum_{n=0}^{\infty}\left(\frac{z+3}{9}\right)^n - \frac{1}{z+3}\right) = -\sum_{n=0}^{\infty} \frac{(z+3)^n}{9^{n+2}} - \underbrace{\frac{1/9}{z+3}}_{\text{the principal part of the Laurent series}}$$

b) As we already established in (a) the series is absolutely convergent for $0 < |z+3| < 9$.

(Q5) a) $f(z) = \dfrac{z-\sin z}{z^4}$.

To find the singularities we use the Taylor series expansion for $\sin z$ as follows

$$\frac{z-\sin z}{z^4} = \frac{1}{z^4}(z-\sin z) = \frac{1}{z^4}\left(z - z + \frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} - \dots\right)$$
$$= \frac{1}{z}\left(\frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \dots\right) = \frac{1}{z}\varphi(z)$$

$\varphi(z) = \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \dots$ is analytic in a neighborhood of $z=0$ (including $z=0$) and $\varphi(0) = \frac{1}{3!} = \frac{1}{6} \neq 0$. Therefore, by Proposition (*) (formulated below) $z=0$ is a simple pole. There are no other singularities.

Proposition (*): $z = z_0$ is a pole of order $m>0$ of $f \Leftrightarrow f$ can be expressed as $f(z) = \dfrac{\varphi(z)}{(z-z_0)^m}$ where $\varphi(z)$ is analytic at a neighborhood of z_0 (and at z_0) and $\varphi(z_0) \neq 0$.

b) The Cauchy Integral formula: If f is analytic on and everywhere inside a simple, closed, positively oriented ~~contour~~ C, and if z_0 is any point inside C, then

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz.$$

$z=0$ is inside the given contour $C = \{z \in \mathbb{C} \mid |z| = 2\}$

From (a) we have

$f(z) = \dfrac{\varphi(z)}{z}$, where $\varphi(z)$ is analytic on and everywhere inside C and $\varphi(0) = \frac{1}{6}$. Therefore, by Cauchy Integral formula we have

$$\frac{1}{6} = \varphi(0) = \frac{1}{2\pi i}\int_{|z|=2} \frac{\varphi(z)}{z}\,dz = \frac{1}{2\pi i}\int_{|z|=2} \frac{1}{z}\cdot\left(\frac{z-\sin z}{z^3}\right)dz,$$

namely

$$\int_{|z|=2} \frac{z-\sin z}{z^4}\,dz = \frac{2\pi i}{6} = \frac{\pi i}{3}.$$

Alternative: We can use the Residue Theorem:

Let f be analytic on and inside the simple, closed, positively oriented contour C, except at a finite number of isolated singularities $z_1, z_2, \dots, z_n$ all inside C. Then: $\displaystyle\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f, z_k)$

From (a) $z=0$ is the only singularity of $f(z)$ (simple pole) and it is inside C. From (a)

$$f(z) = \frac{1}{6}\cdot\frac{1}{z} - \frac{z}{5!} + \frac{z^3}{7!} - \dots$$

Therefore $\quad \operatorname{Res}(f, 0) = \frac{1}{6}$ (the coefficient of $\frac{1}{z-0} = \frac{1}{z}$).

Thus, by the Residue Theorem, we obtain

$$\int_{|z|=2} \frac{z-\sin z}{z^4}\,dz = 2\pi i \operatorname{Res}\left(\frac{z-\sin z}{z^4}, 0\right) = 2\pi i\cdot\frac{1}{6} = \frac{\pi i}{3}.$$