Complex Variables Final Exam (2018) Solution.

(Q1)

a) Note that

$i+i^2+i^3+i^4 = i-1-i+1=0$, namely for any $k\in\mathbb{N}$ $i^k+i^{k+1}+i^{k+2}+i^{k+3}=0$

Therefore, we have

$i+i^2+i^3+i^4+\ldots+i^{97}+i^{98}+i^{99}+i^{100}=0$ and

$i+i^2+i^3+\ldots+i^{100}+i^{101}=i$

$\Rightarrow$ $(i+i^2+i^3+\ldots+i^{101})^5 = i^5$

By De Moivre formula: for any $z\in\mathbb{C}$: $z^n = |z|^n(\cos n\theta + i\sin n\theta)$.

Since $|i|=1$ and $\arg i = \frac{\pi}{2}$ $(+2\pi k)$ we obtain

$$i^5 = 1^5\left(\cos\left(5\cdot\frac{\pi}{2}\right)+i\sin\left(5\cdot\frac{\pi}{2}\right)\right) = \cos\left(\frac{\pi}{2}+2\pi\right)+i\sin\left(\frac{\pi}{2}+2\pi\right)$$

$$= \cos\frac{\pi}{2}+i\sin\frac{\pi}{2} = 0+i\cdot 1 = i$$

b) $z^7=i$, namely $z=\sqrt[7]{i}$. Write i in polar form:

$|i|=1$, $\arg i = \frac{\pi}{2}(+2\pi k)$, therefore $i = 1\left(\cos\left(\frac{\pi}{2}+2\pi k\right)+i\sin\left(\frac{\pi}{2}+2\pi k\right)\right)$

Therefore,

$$\sqrt[7]{i} = 1^{1/7}\left(\cos\left(\frac{\frac{\pi}{2}+2\pi k}{7}\right)+i\sin\left(\frac{\frac{\pi}{2}+2\pi k}{7}\right)\right) \quad k=0,1,2,3,4,5,6:$$

$k=0$ $\quad z_0 = \cos\frac{\pi}{14}+i\sin\frac{\pi}{14}$

$k=1$ $\quad z_1 = \cos\frac{5\pi}{14}+i\sin\frac{5\pi}{14}$

$k=2$ $\quad z_2 = \cos\frac{9\pi}{14}+i\sin\frac{9\pi}{14}$

$k=3$ $\quad z_3 = \cos\frac{13\pi}{14}+i\sin\frac{13\pi}{14}$

$k=4$ $\quad z_4 = \cos\frac{17\pi}{14}+i\sin\frac{17\pi}{14}$

$k=5$ $\quad z_5 = \cos\frac{21\pi}{14}+i\sin\frac{21\pi}{14}$

$k=6$ $\quad z_6 = \cos\frac{25\pi}{14}+i\sin\frac{25\pi}{14}$

(Q2)

a) Let $\sum_{n=0}^{\infty} a_n$, $a_n\in\mathbb{C}$. Consider the limit $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = L$

- If L exists and $L<1$, then $\sum_{n=0}^{\infty} a_n$ converges absolutely
- If L exists and $L>1$, then $\sum_{n=0}^{\infty} a_n$ diverges
- If L does not exist or $L=1$, then the test is inconclusive.

b) The formula for the radius of convergence is

$$R = \lim_{n\to\infty} \left|\frac{b_n}{b_{n+1}}\right| \quad \text{where} \quad \sum_{n=0}^{\infty} b_n z^n, \quad b_n \in \mathbb{C}$$

We apply the Ratio Test to check the convergence of this series:

$$a_n = \frac{n!}{n^n} z^n, \qquad a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}} z^{n+1}$$

Therefore:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\, z^{n+1}}{(n+1)^{n+1}} : \frac{n!\, z^n}{n^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\, z^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{n!\, z^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n+1}{(n+1)^{n+1}} n^n z\right| = \lim_{n\to\infty}\left|\frac{n^n}{(n+1)^n} z\right| = |z| \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n =$$

$$= |z| \lim_{n\to\infty}\left(\frac{1}{\frac{n+1}{n}}\right)^n = |z| \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^n = |z| \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}$$

Since $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ we obtain $\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}$, therefore

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |z|\frac{1}{e}$ and we have

$$|z|\frac{1}{e} < 1 \iff |z| < e,$$

namely, by the Ratio Test, for $|z|<e$ the series converges absolutely.

The radius of convergence: $b_n = \frac{n!}{n^n}$, $b_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$

$$R = \lim_{n\to\infty}\left|\frac{b_n}{b_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n!}{n^n}\cdot\frac{(n+1)^{n+1}}{(n+1)!}\right| = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

(Q3)

a) The given function $f(z) = \frac{2+3z}{z^2+z^4}$ is analytic on the annulus $0<|z|<1$: the denominator vanishes iff $z^2+z^4=0 \iff z^2(1+z^2)=0 \iff z=0$ or $z=\pm i$, which are outside of $0<|z|<1$. To obtain the series we proceed as follows

$$\frac{2+3z}{z^2+z^4} = \frac{2+3z}{z^2}\cdot\frac{1}{1+z^2} = \left(\frac{2}{z^2}+\frac{3}{z}\right)\frac{1}{1-(-z^2)} = \left(\frac{2}{z^2}+\frac{3}{z}\right)\sum_{n=0}^{\infty}(-z^2)^n =$$

$\frac{1}{1-(-z^2)}$ is the sum of geometric series $\sum_{n=0}^{\infty}(-z^2)^n$

$$= \left(\frac{2}{z^2}+\frac{3}{z}\right)\sum_{n=0}^{\infty}(-1)^n z^{2n} = \frac{2}{z^2}+\frac{3}{z} - 2 - 3z + 2z^2 + 3z^3 - \dots$$

(b) $z=0$ is a pole of order 2 since $f(z) = \frac{1}{z^2}\cdot\frac{2+3z}{1+z^2} = \frac{1}{z^2}\cdot\varphi(z)$ and $\varphi(z)$ is analytic at the neighborhood of $z=0$ and $\varphi(0) = 2 \neq 0$.

We have proved the following proposition:

$z = z_0$ is a pole of order $m > 0$ of $f \iff f$ can be expressed as $f(z) = \frac{\varphi(z)}{(z-z_0)^m}$ where $\varphi(z)$ is analytic at a neighborhood of z_0 (and at z_0) and $\varphi(z_0) \neq 0$. Moreover, $\text{Res}(f, z_0) = \frac{\varphi^{(m-1)}(z_0)}{(m-1)!}$.

We apply this proposition with $\varphi(z) = \frac{2+3z}{1+z^2}$ and $m = 2$ and get

$$\text{Res}(f, 0) = \frac{\varphi'(0)}{1!}.$$

$$\varphi'(z) = \frac{3(1+z^2) - 2z(2+3z)}{(1+z^2)^2} \implies \varphi'(0) = 3 \text{ and}$$

$$\text{Res}(f, 0) = \frac{3}{1!} = 3.$$

(Q4)

a) The branch point of $\text{Ln}(z - z_0)$ is $z = z_0$. Since (proved in class) for any $z_1, z_2 \in \mathbb{C} \setminus \{0\}$ $\text{Ln}\frac{z_1}{z_2} = \text{Ln}\, z_1 - \text{Ln}\, z_2$ we get

$$f(z) = (z+2)\,\text{Ln}\left(\frac{z-2}{z+i}\right) + 5 = (z+2)\left(\text{Ln}(z-2) - \text{Ln}(z+i)\right) + 5$$

and the branch points are $z = 2$ and $z = -i$.

b) Since $\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$ we obtain

$$z^6 \cosh\frac{1}{z} = z^6 \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{-2n} = z^6\left(1 + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{4!}\frac{1}{z^4} + \frac{1}{6!}\frac{1}{z^6} + \frac{1}{8!}\frac{1}{z^8} + \dots\right)$$

$$= z^6 + \frac{z^4}{2!} + \frac{z^2}{4!} + \frac{1}{6!} + \frac{1}{8!}\frac{1}{z^2} + \frac{1}{10!}\frac{1}{z^4} + \dots$$

Therefore we see that there is an isolated singularity at $z = 0$. Since the principal part of the series has infinitely many terms, by definition it is an essential singularity

(Q5)

a) Let f be analytic on and inside the simple, closed, positively oriented contour C, except at a finite number of isolated singularities $z_1, .., z_n$ all inside C. Then,

$$\int_C f(z)\,dz = 2\pi i \sum_{K=1}^{n} \text{Res}(f, z_K).$$

b) $$\int_{|z-1| = \frac{5}{2}} \frac{dz}{(z-5)(z+1)^4}$$

The singularities are at $z=5$ and $z=-1$, but $z=5$ is outside of $|z-1| \leq \frac{5}{2}$, thus we can ignore it. First, note that $z=-1$ is a pole of order 4, since

$f(z) = \frac{1}{(z-5)(z+1)^4} = \frac{1}{(z+1)^4}\varphi(z)$, $\varphi(z) = \frac{1}{z-5}$ and $\varphi(z)$ is analytic at a neighborhood of $z=-1$ (and at $z=-1$), $\varphi(-1) = -\frac{1}{6} \neq 0$.

Therefore, since (for $m=4$)

$$\operatorname{Res}(f,-1) = \frac{\varphi'''(-1)}{3!}$$

we obtain:

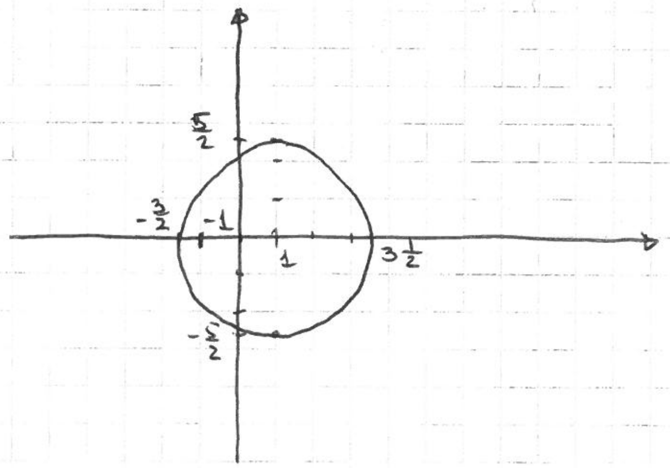
$$\varphi'(z) = -\frac{1}{(z-5)^2}, \quad \varphi''(z) = \frac{2}{(z-5)^3}, \quad \varphi'''(z) = -\frac{6}{(z-5)^4} \Rightarrow \varphi'''(-1) = -\frac{1}{6^3}$$

and we have

$$\operatorname{Res}(f,-1) = -\frac{1}{6^3}\cdot\frac{1}{3!} = -\frac{1}{6^3}\cdot\frac{1}{6} = -\frac{1}{6^4}.$$

Therefore, by the Residue theorem

$$\int_{|z-1|=\frac{5}{2}} \frac{dz}{(z-5)(z+1)^4} = 2\pi i \operatorname{Res}\left(\frac{1}{(z-5)(z+1)^4}, -1\right) = 2\pi i\left(-\frac{1}{6^4}\right)$$



Alternatively: We have a simple pole at $z=5$ and a pole of order 4 at $z=-1$. By the definition of the contour $z=-1$ lies inside the conto[ur] and $z=5$ does not lies inside C. For $\varphi(z) = \frac{1}{z-5}$ by extended Cauch[y] Integral formula we have for $n=3$

$$\varphi^{(3)}(-1) = \frac{3!}{2\pi i}\int_{|z-1|=\frac{5}{2}} \frac{\varphi(z)}{(z+1)^4}dz$$

$$\Rightarrow \int_{|z-1|=\frac{5}{2}} \frac{\varphi(z)}{(z+1)^4}dz = \frac{2\pi i}{3!}\varphi^{(3)}(-1) = \frac{2\pi i}{3!}\left(-\frac{1}{6^3}\right) = 2\pi i\left(-\frac{1}{6^4}\right).$$