Problem Set 3 - Solutions.

1) a) By Def 3.2: need to show that for any $\varepsilon > 0$ there exists $R = R(\varepsilon) > 0$ such that if $|z| > R$, then $\left|\frac{3z-8i}{4z+3} - \frac{3}{4}\right| < \varepsilon$.

Choose $R = \frac{\sqrt{1105}}{16\varepsilon} + 1$. We have

$$(*)\ \left|\frac{3z-8i}{4z+3} - \frac{3}{4}\right| = \left|\frac{12z-32i-12z-9}{4(4z+3)}\right| = \frac{|-32i-9|}{4|4z+3|} = \frac{\sqrt{32^2+9^2}}{4|4z+3|} = \frac{\sqrt{1105}}{4|4z+3|}$$

By reverse triangle inequality we get

$$|4z+3| \geq |4|z|-3| \underset{|z|>R}{\geq} |4R-3| > |4R-4| \underset{\text{assume } R\geq 1}{=} 4(R-1) =$$

$$\underset{\text{choice of } R}{=} 4\left(\frac{\sqrt{1105}}{16\varepsilon} + 1 - 1\right) = \frac{\sqrt{1105}}{4\varepsilon}$$

Thus:

$$(*) = \frac{\sqrt{1105}}{4|4z+3|} < \frac{\sqrt{1105}}{4\cdot\frac{\sqrt{1105}}{4\varepsilon}} = \varepsilon$$

$\blacksquare$

b) By Def 3.3: need to show that for any $R > 0$ there exists $\delta = \delta(R) > 0$ such that if $0 < |z+8| < \delta$, then $\left|\frac{1}{z+8}\right| > R$.

Choose $\delta = \frac{1}{R}$ - then, whenever $0 < |z+8| < \delta = \frac{1}{R}$ we get $\left|\frac{1}{z+8}\right| > R$.

c) By Def 3.4: need to show that for any $R > 0$ there exists $S = S(R) > 0$ such that if $|z| > S$, then $\left|\frac{z^4-2z}{3z^3+4z^2}\right| > R$

Choose $S = 7R$

$$\left|\frac{z^4-2z}{3z^3+4z^2}\right| = \left|\frac{z^3-2}{3z^2+4z}\right| = \left|\frac{z^3\left(1-\frac{2}{z^3}\right)}{z^2\left(3+\frac{4}{z}\right)}\right| = |z|\,\frac{\left|1-\frac{2}{z^3}\right|}{\left|3+\frac{4}{z}\right|}\ (*)$$

Assume we have shown that $\frac{|1-2/z^3|}{|3+\frac{4}{z}|} > \frac{1}{7}$. Then

$$(*) > |z|\frac{1}{7} \underset{\text{choice of } S}{>} \frac{1}{7}7R = R$$ and this finishes the proof.

Let us show $(*)$. By reverse triangle inequality

$$\left|1-\frac{2}{z^3}\right| \geq \left|1-\frac{2}{|z|^3}\right| \underset{\text{since } z\to\infty \text{ we can assume } |z|>1}{\geq} \left|1-\frac{2}{1}\right| = 1$$

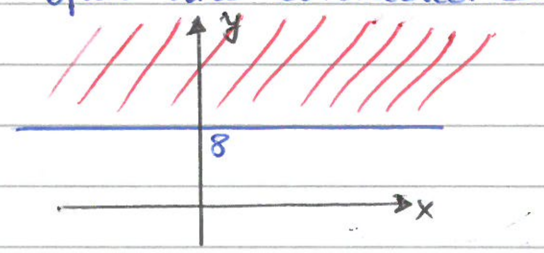
On the other hand, by triangle inequality

$$\left|3+\frac{4}{z}\right| \le 3+\frac{4}{|z|} \underset{|z|>1}{<} 3+\frac{4}{1} = 7$$

$$\Rightarrow \frac{\left|1-\frac{2}{z^3}\right|}{\left|3+\frac{4}{z}\right|} > \frac{1}{7}.$$

2) a) $\operatorname{Im} z > 8$ is a domain:

$\{\operatorname{Im} z > 8\} = \{(x,y) : y > 8\}$ - open and connected. Its boundary is $\{(x,y) : y = 8\}$



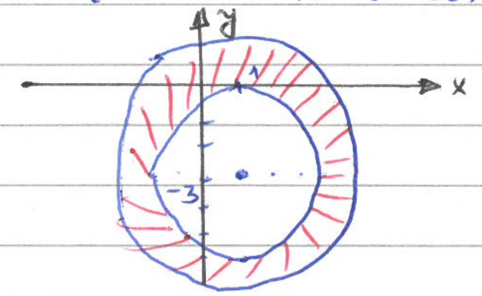
b) $3 < |z-1+3i| < 5$ is a domain:

$|z-(1-3i)| = 3$ - circle of radius 3 centered at $z = 1-3i$

$|z-(1-3i)| = 5$ - circle of radius 5 centered at $z = 1-3i$

It is open and connected (thus - domain), its boundary is

$\{z \in \mathbb{C} : |z-(1-3i)| = 3,\ |z-(1-3i)| = 5\}$

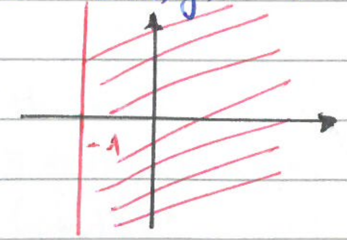


c) $|z+2| \ge |z|$:

$|z+2| = \sqrt{(x+2)^2+y^2} \ge \sqrt{x^2+y^2} = |z| \Leftrightarrow (x+2)^2+y^2 \ge x^2+y^2$

$\Leftrightarrow (x+2)^2 \ge x^2 \Leftrightarrow 4x+4 \ge 0 \Leftrightarrow x \ge -1$

This is not a domain since it includes all its boundary points $\{(x,y) : x = -1\}$, thus it is a closed (connected) set



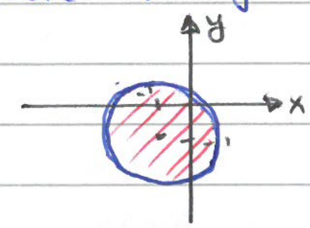
d) First, let us see that this is meaningful expression, namely let us check that $z\bar{z} + (1-i)z + (1+i)\bar{z} \in \mathbb{R}$ (otherwise it is meaningless!)

$z\bar z + (1-i)z + (1+i)\bar z = z\bar z + \underbrace{z+\bar z}_{2\operatorname{Re} z} - \underbrace{i(z-\bar z)}_{2\operatorname{Im} z} = |z|^2 + 2\operatorname{Re} z +$
$+ 2\operatorname{Im} z = (\sqrt{x^2+y^2})^2 + 2x + 2y \in \mathbb{R}$

Thus, we get:

$$x^2+y^2+2x+2y<0 \underset{\text{complete square in } x \text{ and } y}{\iff} (x+1)^2+(y+1)^2<2$$

This equation describes the interior of the circle centered at $(-1,-1)$ of radius $\sqrt{2}$ without the boundary $\Rightarrow$ this set is open and connected $\Rightarrow$ it is a domain. Its boundary is the circle $\{(x,y) : (x+1)^2+(y+1)^2 = 2\}$



3) a) Let us check whether the limit $\lim_{z\to i} \frac{(z^2+1)(z^2-3iz+2)}{z^2+(1-i)z-i}$ exists. Note: $z=i$ is a root of the denominator:

$$(z^2+z(1-i)-i)|_{z=i} = i^2+i(1-i)-i = i^2+i-i^2-i = 0$$

Note also, that $z=i$ is a root of the numerator:

$$((z^2+1)(z^2-3iz+2))|_{z=i} = (i^2+1)(i^2-3i^2+2)$$
$$= (-1+1)(-1+3+2) = 0$$

Namely, the numerator and the denominator have a common factor $z=i$:

$$z^2+1 = (z-i)(z+i)$$
$$z^2+z(1-i)-i = (z-i)(z+1)$$

$$\Rightarrow \frac{(z^2+1)(z^2-3iz+2)}{z^2+z(1-i)-i} = \frac{(z-i)(z+i)(z^2-3iz+2)}{(z-i)(z+1)} \underset{z\neq i \Rightarrow \text{ we can factorize}}{=} \frac{(z+i)(z^2-3iz+2)}{(z+1)}$$

Now the denominator $(z+1)|_{z=i} = 1+i \neq 0$, thus applying Prop 5.1: $g(z) = z+1$ is continuous at $z=i$: $\lim_{z\to i} g(z) = i+1 \neq 0$
$f(z) = (z+i)(z^2-3iz+2)$ is continuous at $z=i$ as well:

$$\lim_{z\to i} f(z) = 2i(-1+3+2) = 8i$$

Thus, by Prop 5.1 $\frac{f}{g}$ is continuous at $z=i$, and:

$$\lim_{z\to i} \frac{f(z)}{g(z)} = \frac{8i}{1+i} = \frac{8i}{1+i}\cdot\frac{1-i}{1-i} = \frac{8i+8}{2} = 4+4i$$

b) $f(z) = \begin{cases} \bar{z}/z & z\neq 0 \\ 0 & z=0 \end{cases}$ is not continuous at $z_0=0$ but continuous for any other z_0.

We need to show 2 things:

1) $\lim\limits_{z\to 0} f(z)$ does not exist (or does not equal to 0)

2) $\lim\limits_{z\to z_0\neq 0} f(z) = f(z_0) = \bar{z}_0/z_0$

1) let $z=re^{i\theta}$. If $z\to 0 \Rightarrow r\to 0$, and

$$\lim_{z\to 0} \bar{z}/z = \lim_{r\to 0} re^{-i\theta}/re^{i\theta} = \lim_{r\to 0} e^{-2i\theta} = e^{-2i\theta}$$

Thus, the limit takes different values for different $\theta \Rightarrow$ does not exist (since it is not unique!)

2) Assume $z_0 = r_0 e^{i\theta_0}\neq 0$. Then $z\to z_0$ means $(r,\theta)\to(r_0,\theta_0)$, and we have

$$\lim_{z\to z_0} \bar{z}/z = \lim_{(r,\theta)\to(r_0,\theta_0)} re^{-i\theta}/re^{i\theta} = \lim_{(r,\theta)\to(r_0,\theta_0)} e^{-2i\theta} = e^{-2i\theta_0} = f(z_0)$$

Thus, the limit exists and it is equal to the value of the function f at $z_0 \Rightarrow f$ is continuous at every $z_0\neq 0$.

c) $f(z) = \begin{cases} (\mathrm{Re}\,z)^4/|z|^3 & z\neq 0 \\ 0 & z=0 \end{cases}$ is continuous for all $z_0\in\mathbb{C}$.

We need to show 2 things:

1) $\lim\limits_{z\to 0} f(z) = f(0) = 0$

2) $\lim\limits_{z\to z_0} f(z) = f(z_0) = (\mathrm{Re}\,z_0)^4/|z_0|^3$

let $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$. Then $|z|^3 = r^3$ and $\mathrm{Re}\,z = r\cos\theta$

1) $\lim\limits_{z\to 0} f(z) = \lim\limits_{r\to 0} (r\cos\theta)^4/r^3 = \lim\limits_{r\to 0} r^4\cos^4\theta/r^3 = \lim\limits_{r\to 0} r\cos^4\theta = 0$

$\Rightarrow f$ is continuous at $z=0$

2) $\lim\limits_{z\to z_0\neq 0} f(z) = \lim\limits_{(r,\theta)\to(r_0,\theta_0)} r^4\cos^4\theta/r^3 = \lim\limits_{(r,\theta)\to(r_0,\theta_0)} r\cos^4\theta = r_0\cos^4\theta_0 = f(z_0)$

$\Rightarrow f$ is continuous at $z=z_0\,(\neq 0)$.

$\Rightarrow f$ is continuous for all $z_0\in\mathbb{C}$.

4) a) i) We use the Chain Rule:

$g(z) = z^4 \quad f(z) = 8z^5 - 3i$

$(g \circ f)'(z) = g'(f(z))\, f'(z) = 4(8z^5-3i)^3\, 8 \cdot 5 z^4 = 160(8z^5-3i)^3 z^4$

ii) We apply Prop 6.1 and Ex 8.2:

$$\left(\frac{z^3+1}{8i-3-z}\right)' = \frac{(8i-3-z)\,3z^2 - (z^3+1)(-1)}{(8i-3-z)^2} = \frac{-2z^3 - z^2(9-24i)+1}{(8i-3-z)^2}$$

iii) We use Def 6.1:

$$\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \lim_{\Delta z \to 0}\left[\frac{(z_0+\Delta z)^3 - 3(z_0+\Delta z)^2 + 6(z_0+\Delta z) - 8}{\Delta z}\right.$$

$$\left. - \frac{z_0^3 - 3z_0^2 + 6z_0 - 8}{\Delta z}\right] = \lim_{\Delta z \to 0}\left[\frac{z_0^3 + 3z_0^2\Delta z + 3z_0\Delta z^2 + \Delta z^3 - 3z_0^2 - 6z_0\Delta z + \Delta z^2}{\Delta z}\right.$$

$$\left. + \frac{6z_0 + 6\Delta z - 8 - z_0^3 + 3z_0^2 - 6z_0 + 8}{\Delta z}\right] =$$

$$= \lim_{\Delta z \to 0} \frac{3z_0^2\Delta z + 3z_0\Delta z^2 + \Delta z^3 - 6z_0\Delta z + \Delta z^2 + 6\Delta z}{\Delta z}$$

$$= \lim_{\Delta z \to 0} (3z_0^2 + 3z_0\Delta z + \Delta z^2 - 6z_0 + \Delta z + 6) = 3z_0^2 - 6z_0 + 6$$

$\Rightarrow f'(z) = 3z^2 - 6z + 6$ for all $z \in \mathbb{C}$

b) i) $f(x,y) = x^4 - 4xy^3 + i(y^4 - 6x^2)$

$\Rightarrow u(x,y) = x^4 - 4xy^3 \quad v(x,y) = y^4 - 6x^2$ – u, v are differentiable everywhere. Let us check where CR equations hold

$$\frac{\partial u}{\partial x} = 4x^3 - 4y^3 \qquad \frac{\partial v}{\partial y} = 4y^3$$

$$\frac{\partial u}{\partial y} = -12xy^2 \qquad \frac{\partial v}{\partial x} = -12x$$

Thus:

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \iff \begin{cases} 4x^3 - 4y^3 = 4y^3 \\ -12xy^2 = 12x \end{cases} \iff \begin{cases} x^3 - y^3 = y^3 \\ -xy^2 = x \end{cases}$$

If $x = 0 \Rightarrow$ from $x^3 - y^3 = y^3$ we get $-y^3 = y^3 \iff y = 0$.
If $x \neq 0 \Rightarrow$ from the second equation $-y^2 = 1$ and since $y \in \mathbb{R}$ this equation does not hold for any y. Thus, f is differentiable only at $(0,0)$ and $f'(0) = 0$.

ii) $f(z) = z/\bar{z}$, $z \neq 0$. First, let us write representation

$$f(x,y) = u(x,y) + iv(x,y).$$

Let $z = x+iy$. Then $\bar{z} = x - iy$, and

$$\frac{z}{\bar z} = \frac{x+iy}{x-iy} = \frac{x+iy}{x-iy}\cdot\frac{x+iy}{x+iy} = \frac{(x+iy)^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2} + i\,\frac{2xy}{x^2+y^2}$$

$$\Rightarrow u(x,y) = \frac{x^2-y^2}{x^2+y^2} \qquad v(x,y) = \frac{2xy}{x^2+y^2}$$

Thus:

$$\frac{\partial u}{\partial x} = \frac{4xy^2}{(x^2+y^2)^2} \qquad \frac{\partial v}{\partial y} = \frac{2x^3-2xy^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = -\frac{4yx^2}{(x^2+y^2)^2} \qquad \frac{\partial v}{\partial x} = \frac{2y^3-2x^2y}{(x^2+y^2)^2}$$

$$\Rightarrow \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \Leftrightarrow \begin{cases} 4xy^2 = 2x^3 - 2xy^2 \\ 4yx^2 = 2y^3 - 2x^2y \end{cases} \Leftrightarrow \begin{cases} 2x(x^2-3y^2) = 0 \\ y(2y^2-6x^2) = 0 \end{cases}$$

$\Rightarrow$ the unique solution of this system is $(x,y) = (0,0)$. But $z \neq 0$ $\Rightarrow$ f is not differentiable at any point.

iii) $f(z) = 8e^{3iz^2}$

Let $z = x+iy$. Then $z^2 = x^2-y^2+i\,2xy$. Thus:

$$8e^{3iz^2} = 8e^{3i(x^2-y^2+i2xy)} = 8e^{3i(x^2-y^2)-6xy} =$$

$$= 8e^{-6xy}e^{i(3(x^2-y^2))} = 8e^{-6xy}\left(\cos(3(x^2-y^2)) + i\sin(3(x^2-y^2))\right)$$

$$\Rightarrow u(x,y) = 8e^{-6xy}\cos(3(x^2-y^2)) \qquad v(x,y) = 8e^{-6xy}\sin(3(x^2-y^2))$$

u and v are differentiable everywhere. Let us check for which (x,y) CR equations hold.

$$\frac{\partial u}{\partial x} = -48ye^{-6xy}\cos(3(x^2-y^2)) + 8e^{-6xy}(-\sin(3(x^2-y^2))\cdot 6x)$$

$$= -48ye^{-6xy}\cos(3(x^2-y^2)) - 48xe^{-6xy}\sin(3(x^2-y^2))$$

$$\frac{\partial u}{\partial y} = -48xe^{-6xy}\cos(3(x^2-y^2)) + 8e^{-6xy}(-\sin(3(x^2-y^2))(-6y))$$

$$= -48xe^{-6xy}\cos(3(x^2-y^2)) + 48ye^{-6xy}\sin(3(x^2-y^2))$$

$$\frac{\partial v}{\partial x} = -48ye^{-6xy}\sin(3(x^2-y^2)) + 8e^{-6xy}\cos(3(x^2-y^2))\,6x$$

$$= -48ye^{-6xy}\sin(3(x^2-y^2)) + 48xe^{-6xy}\cos(3(x^2-y^2))$$

$$\frac{\partial v}{\partial y} = -48xe^{-6xy}\sin(3(x^2-y^2)) + 8e^{-6xy}\cos(3(x^2-y^2))(-6y)$$

$$= -48xe^{-6xy}\sin(3(x^2-y^2)) - 48ye^{-6xy}\cos(3(x^2-y^2))$$

Thus, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ for all (x,y). Thus, since u and v are differentiable everywhere and CR equations hold everywhere we conclude that f is differentiable everywhere, and

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -48ye^{-6xy}\left(\cos(3(x^2-y^2)) + i\sin(3(x^2-y^2))\right)$$
$$-48xe^{-6xy}\left(\sin(3(x^2-y^2)) - i\cos(3(x^2-y^2))\right)$$

$$= -48y\, e^{-6xy} e^{i3(x^2-y^2)} + 48ix\, e^{-6xy} e^{i3(x^2-y^2)} =$$

since $\sin(3(x^2-y^2)) - i\cos(3(x^2-y^2)) = -i\left(\cos(3(x^2-y^2)) + i\sin(3(x^2-y^2))\right)$
$= -ie^{i3(x^2-y^2)}$

$$= -48y\, e^{i(3(x^2-y^2)+i6xy)} + 48ix\, e^{i(3(x^2-y^2)+i6xy)}$$
$$= -48y\, e^{i3z^2} + 48xi e^{i3z^2} = 48ie^{i3z^2}(x+iy) = 48ize^{i3z^2}$$

5) Let $z = re^{i\theta} = r\cos\theta + ir\sin\theta$. Then, if $z = x+iy$:

$$x = r\cos\theta \qquad y = r\sin\theta$$

Namely, x and y are functions of r and θ.

If $f(z) = u(x(r,\theta), y(r,\theta)) + iv(x(r,\theta), y(r,\theta)) = \psi(r,\theta) + i\varphi(r,\theta)$, then applying the Chain Rule we get

$$\frac{\partial \psi}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$$

$$\frac{\partial \psi}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y} r\cos\theta$$

$$\frac{\partial \varphi}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y} r\cos\theta$$

$$\frac{\partial \varphi}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta$$

By the usual CR equations: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, thus

$$\frac{\partial \varphi}{\partial \theta} = -\frac{\partial u}{\partial y}(-r\sin\theta) + \frac{\partial u}{\partial x} r\cos\theta = r\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right) = r\frac{\partial \psi}{\partial r}$$

Namely, $\frac{\partial \psi}{\partial r} = \frac{1}{r}\frac{\partial \varphi}{\partial \theta}$

$$\frac{\partial \varphi}{\partial r} = -\frac{\partial u}{\partial y}\cos\theta + \frac{\partial u}{\partial x}\sin\theta = -\left[-\frac{\partial u}{\partial x}\sin\theta + \frac{\partial u}{\partial y}\cos\theta\right] = -\frac{1}{r}\left[\frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y} r\cos\theta\right] = -\frac{1}{r}\frac{\partial \psi}{\partial \theta},$$

namely, $\frac{\partial \psi}{\partial \theta} = -r\frac{\partial \varphi}{\partial r}$

To prove that $f'(z) = e^{-i\theta}\left(\frac{\partial \psi}{\partial r} + i\frac{\partial \varphi}{\partial r}\right)$ we use that $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

We need to express $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}$ in terms of $\frac{\partial \psi}{\partial r}, \frac{\partial \varphi}{\partial r}$

Note: $\begin{cases} \cos\theta\frac{\partial \psi}{\partial r} = \frac{\partial u}{\partial x}\cos^2\theta + \frac{\partial u}{\partial y}\cos\theta\sin\theta \\ \sin\theta\frac{\partial \varphi}{\partial r} = \frac{\partial u}{\partial x}\sin^2\theta - \frac{\partial u}{\partial y}\cos\theta\sin\theta \end{cases} \Rightarrow \cos\theta\frac{\partial \psi}{\partial r} + \sin\theta\frac{\partial \varphi}{\partial r} = \frac{\partial u}{\partial x}$

In the same way: $\begin{cases} \sin\theta\frac{\partial \psi}{\partial r} = \frac{\partial u}{\partial x}\cos\theta\sin\theta + \frac{\partial u}{\partial y}\sin^2\theta \\ -\cos\theta\frac{\partial \varphi}{\partial r} = -\frac{\partial u}{\partial x}\cos\theta\sin\theta + \frac{\partial u}{\partial y}\cos^2\theta \end{cases}$

$$\Rightarrow \sin\theta\frac{\partial \psi}{\partial r} - \cos\theta\frac{\partial \varphi}{\partial r} = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow \frac{\partial v}{\partial x} = \cos\theta\frac{\partial \varphi}{\partial r} - \sin\theta\frac{\partial \psi}{\partial r}$$

$$\Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \cos\theta\frac{\partial \psi}{\partial r} + \sin\theta\frac{\partial \varphi}{\partial r} + i\cos\theta\frac{\partial \varphi}{\partial r} - i\sin\theta\frac{\partial \psi}{\partial r} =$$
$$= \frac{\partial \psi}{\partial r}(\cos\theta - i\sin\theta) + \frac{\partial \varphi}{\partial r}(\sin\theta + i\cos\theta) = \frac{\partial \psi}{\partial r}(\cos\theta - i\sin\theta) + i\frac{\partial \varphi}{\partial r}(\cos\theta$$
$$+ \tfrac{1}{i}\sin\theta) = \frac{\partial \psi}{\partial r}(\cos\theta - i\sin\theta) + i\frac{\partial \varphi}{\partial r}(\cos\theta - i\sin\theta) = e^{-i\theta}\left(\frac{\partial \psi}{\partial r} + i\frac{\partial \varphi}{\partial r}\right)$$