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Problem Set 3.

1) Using Def 3.2, 3.3, and 3.4 prove that

a) $\lim_{z\to\infty} \frac{3z-8i}{4z+3} = \frac{3}{4}$

b) $\lim_{z\to -8} \frac{1}{z+8} = \infty$

c) $\lim_{z\to\infty} \frac{z^4-2z}{3z^3+4z^2} = \infty$

2) Which of the following sets is a domain? For each set determine what is its boundary (and sketch it on the plane)

a) $\operatorname{Im} z > 8$ b) $3 < |z-1+3i| < 5$ c) $|z+2| \geq |z|$

d) $z\bar{z} + (1-i)z + (1+i)\bar{z} < 0$ (why this is meaningful?)

3) a) Is $f(z) = \frac{(z^2+1)(z^2-3iz+2)}{z^2+z(1-i)-i}$ continuous at $z=i$?

b) Prove that $f(z) = \begin{cases} \bar{z}/z & z \neq 0 \\ 0 & z = 0 \end{cases}$ is not continuous at $z_0=0$, but continuous for all other z_0

c) Prove that $f(z) = \begin{cases} (\operatorname{Re} z)^4/|z|^3 & z \neq 0 \\ 0 & z=0 \end{cases}$ is continuous for all $z_0 \in \mathbb{C}$

4) a) Compute the derivative of the following functions: Justify all the steps

i) $(8z^5-3i)^4$ ii) $\frac{z^3+1}{8i-3-z}$ iii) z^3-3z^2+6z-8

b) Using Cauchy-Riemann equations find where the following functions are differentiable:

i) $f(x,y) = x^4 - 4xy^3 + i(y^4 - 6x^2)$

ii) $f(z) = z/\bar{z}$ $(z \neq 0)$ {Represent in the form $u(x,y)+iv(x,y)$)

iii) $f(z) = 8e^{3iz^2}$ (Represent in the form $u(x,y)+iv(x,y)$)

5) Using the Chain Rule prove the Polar form of Cauchy-Riemann equations: Let $z = re^{i\theta}$, $f(z) = \psi(r,\theta) + i\varphi(r,\theta)$

If f is differentiable at z_0, then $\frac{\partial\psi}{\partial r}, \frac{\partial\psi}{\partial\theta}, \frac{\partial\varphi}{\partial r}, \frac{\partial\varphi}{\partial\theta}$ exist, and $\frac{\partial\psi}{\partial r} = \frac{1}{r}\frac{\partial\varphi}{\partial\theta}$ and $\frac{\partial\psi}{\partial\theta} = -r\frac{\partial\varphi}{\partial r}$.

Moreover: $f'(z) = e^{-i\theta}\left(\frac{\partial\psi}{\partial r} + i\frac{\partial\varphi}{\partial r}\right)$