MTH5103 Complex Variables

Week 3 Practice Exercies

These exercises are for your daily practice.

1. Using the definition of a limit, prove that for any $z_0 \neq 0$, $\lim_{z \to z_0} \frac{\bar{z}}{z} = \frac{\bar{z}_0}{z_0}$.

2. Prove each of the three parts of Proposition 1 in the Week 3 Lecture, namely if $f(z), g(z)$ have limits w_0, w_1 as $z \to z_0$, then

   (a) $f(z) + g(z)$ has limit $w_0 + w_1$ as $z \to z_0$

   (b) $f(z)g(z)$ has limit $w_0 w_1$ as $z \to z_0$ (*Hint: try adding and subtracting $f(z)w_1$ to your expression*)

   (c) $\frac{1}{f(z)}$ has limit $\frac{1}{w_0}$ as $z \to z_0$, provided $w_0 \neq 0$. (*Hint: you may need to estimate the value of $|f(z)|$ in this part.*)

3. Prove that

$$f(z) = \begin{cases} \frac{\bar{z}}{z} & \text{for } z \neq 0 \\ 0 & \text{for } z = 0 \end{cases} . \tag{1}$$

   is continuous at all $z_0 \neq 0$.

4. Prove that the function defined by

$$f(z) = \begin{cases} \frac{z^2}{|z|} & \text{for } z \neq 0 \\ 0 & \text{for } z = 0 \end{cases} \tag{2}$$

   is continuous *for all* $z_0 \in \mathbb{C}$. Note that to check this, you need to show that $\lim_{z \to 0} \frac{z^2}{|z|} = 0$ and also that for $z_0 \neq 0$, $\lim_{z \to z_0} \frac{z^2}{|z|} = \frac{z_0^2}{|z_0|}$.

5. What can you say about the limit of $f(z) = \frac{z^2}{|z|^2}$ as z approaches 0?

6. Go through the proof to Proposition 4 in the Week 3 Lecture carefully. Be sure that you understand all of the $\epsilon - \delta$ details!

7. Prove that $\lim_{z \to -1} \frac{iz + 3}{z + 1} = \infty$ by calculating $\lim_{z \to -1} \frac{z + 1}{iz + 3}$ to be zero.

8. Notice that the proof of Proposition 7 in the Week 3 Lecture also holds for real functions. Can you think of a function which is continuous everywhere, but differentiable nowhere?