Week 3 Lecture 7

We have proved:

$$\lim_{z\to z_0} f(z) = u_0 + iv_0 \iff \lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \text{ and } \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$$

Cor 3.1: If $\lim_{z\to z_0} f(z) = w_0$, then $\lim_{z\to z_0} |f(z)| = |w_0|$. In addition, if $w_0 \neq 0$ and $\arg w_0$ is chosen in the interval $[-\pi, \pi]$, then $\lim_{z\to z_0} \arg f(z) = \arg w_0$.

We extend the definition of limit to include the point at ∞:

Def 3.2: We say that $\lim_{z\to\infty} f(z) = w_0$ if for any (given) $\varepsilon > 0$ there exists $R = R(\varepsilon) > 0$ such that for any $|z| > R$ $|f(z) - w_0| < \varepsilon$

Def 3.3: We say that $\lim_{z\to z_0} f(z) = \infty$ if for any (given) $R > 0$ there exists $\delta = \delta(R) > 0$ such that if $0 < |z - z_0| < \delta$, then $|f(z)| > R$

Def 3.4: We say that $\lim_{z\to\infty} f(z) = \infty$ if for any $R > 0$ there exists $S = S(R) > 0$ such that if $|z| > S$, then $|f(z)| > R$

Ex 5.5: Prove that a) $\lim_{z\to i} \frac{1}{z-i} = \infty$ b) $\lim_{z\to\infty} \frac{1}{z} = 0$

Sol: a) By Def 3.3 we need to find $\delta = \delta(R) > 0$ for a given $R > 0$: If $\left|\frac{1}{z-i}\right| > R$, then $|z - i| < \frac{1}{R}$, thus the choice $\delta = \frac{1}{R}$ finishes the proof

b) By Def 3.2 we need to find $R > 0$ for a given $\varepsilon > 0$. If $\left|\frac{1}{z} - 0\right| = \left|\frac{1}{z}\right| < \varepsilon$, then $|z| > \frac{1}{\varepsilon}$, choose $R = \frac{1}{\varepsilon}$ and finish.

Ex 5.6: Prove that $\lim_{z\to\infty} \frac{z^2+1}{z+2i} = \infty$

Sol: By Def 3.4 we need to find $S = S(R) > 0$ for a given $R > 0$. If $\left|\frac{z^2+1}{z+2i}\right| > R$, then

$$\left|\frac{z^2+1}{z+2i}\right| = \left|\frac{z^2\left(1+\frac{1}{z^2}\right)}{z\left(1+\frac{2i}{z}\right)}\right| = |z| \left|\frac{1+\frac{1}{z^2}}{1+\frac{2i}{z}}\right| > R$$

Let us show that $\left|\frac{1+\frac{1}{z^2}}{1+\frac{2i}{z}}\right| < 2$, then

$$R < \left|\frac{z^2+1}{z+2i}\right| = |z| \left|\frac{1+\frac{1}{z^2}}{1+\frac{2i}{z}}\right| < 2|z| \Rightarrow \text{take } S = \frac{R}{2} \text{ and finish.}$$

Since $z \to \infty$ we can assume that $|z| > 1$. Thus

$$(1)\ \left|1 + \frac{1}{z^2}\right| \underset{\text{triangle inequality}}{\leq} 1 + \frac{1}{|z|^2} \leq 1 + 1 = 2$$

$$\left|1+\frac{2i}{z}\right| \underset{\text{reverse triangle inequality}}{\geq} \left|1-\left|\frac{2i}{z}\right|\right| = \left|1-\frac{|2i|}{|z|}\right| = \left|1-\frac{2}{|z|}\right| \underset{|z|>1}{\geq} |1-2| = 1.$$

Thus (2) $\frac{1}{|1+\frac{2i}{z}|} < 1$ and we get

$$\left|\frac{1+\frac{1}{z^2}}{1+\frac{2i}{z}}\right| = \frac{|1+\frac{1}{z^2}|}{|1+\frac{2i}{z}|} \underset{(1)+(2)}{<} \frac{2}{1} = 2$$

Alternatively: By Prop 3.1 2)

$$\lim_{z\to\infty}\left|\frac{z^2+1}{z+2i}\right| = \lim_{z\to\infty}|z|\left|\frac{1+\frac{1}{z^2}}{1+\frac{2i}{z}}\right| = \lim_{z\to\infty}|z| \lim_{z\to\infty}\left|1+\frac{1}{z^2}\right| \lim_{z\to\infty}\frac{1}{|1+\frac{z}{2i}|}$$

By Ex 5.5 b):

$$\lim_{z\to\infty}\left|1+\frac{1}{z^2}\right| = 1 \quad \lim_{z\to\infty}\frac{1}{|1+\frac{2i}{z}|} = 1 \Rightarrow \lim_{z\to\infty}\left|\frac{z^2+1}{z+2i}\right| = \infty$$

Our next subject is continuity of complex functions. First, we need several definitions of sets in the complex plane.

Sets in $\mathbb{C}$.

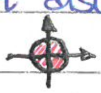
Def 4.1: A subset $S \subseteq \mathbb{C}$ is open if for each $z_0 \in S$ there exists $r>0$ such that the disc $\{z\in\mathbb{C} \mid |z-z_0|<r\} \subseteq S$ is a subset of S

Def 4.2: An open set is connected if it is not a disjoint union of two non-empty open sets {OR: If any two points inside can be connected with a continuous line which is contained in the set}.

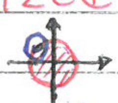
Rmk: In $\mathbb{C}$ these two definitions (in Def 4.2) are equivalent, but it is not always the case. The second - path connected - always implies the first, but there are complicated examples of sets that are connected but not path connected. OR means: For any points z, z' in the set S there exists a continuous function $p: [0,1] \to S$ such that $p(0) = z$, $p(1) = z'$.

Def 4.3: A connected open subset $\Omega \subseteq \mathbb{C}$ is called a domain.

Ex 6.1: a) The interior of the unit disc $\{z\in\mathbb{C} \mid |z|<1\}$ is a domain - it is open and connected.



b) The unit disc with the boundary $\{z\in\mathbb{C} \mid |z|\leq 1\}$ is connected but not open $\Rightarrow$ it is not a domain



Take any point on the boundary $|z|=1$. Then, any disc around this point is not contained in the unit disc regardless how small its radius

c) $\{z\in\mathbb{C} \mid |z|<1\} \cup \{z\in\mathbb{C} \mid |z-3|<1\}$ - open, but not connected

- it is a union of 2 non-empty disjoint open sets

Def 4.4: Let $S\subseteq\mathbb{C}$. $z_0\in S$ is an interior point of S if there exists $r>0$ such that the disc $\{z\in\mathbb{C} \mid |z-z_0|<r\}\subseteq S$

$z_1\in S$ is a boundary point of S if every open disc around it, $\{z\in\mathbb{C} \mid |z-z_1|<r\}$ for any $r>0$ contains points from S and not from S ($\mathbb{C}\setminus S$). The collection of boundary points of S is called the boundary of S and denoted by ∂S.

An equivalent definition of an open set:

Def $4.1\frac{1}{2}$: $S\subseteq\mathbb{C}$ is open if it contains only interior points

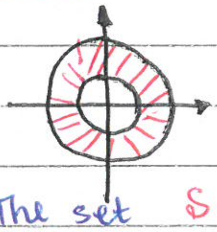
Def 4.5: A set $S\subseteq\mathbb{C}$ is closed if it contains all its boundary points.

Let $S\subseteq\mathbb{C}$ be an open set. If we add to S all its boundary points we always will get a closed set!

The closure of S: $\bar{S} = S\cup\partial S$ is a closed set.

Rmk: There exist sets that are neither open nor closed!

Ex 6.2: 1) The set $S=\{z\in\mathbb{C} \mid 1<|z|<4\}$ is an open set



Its boundary consists of two parts:

$\partial S=\{z\in\mathbb{C} \mid |z|=1, |z|=4\}$ and its closure is

$\bar{S} = S\cup\partial S = \{z\in\mathbb{C} \mid 1\le|z|\le4\}$

2) The set $S=\{z\in\mathbb{C} \mid 1\le|z|<4\}$ neither open nor closed: it contains only part of its boundary points $\{z\in\mathbb{C} \mid |z|=1\}$ but not all the boundary points, as $\{z\in\mathbb{C} \mid |z|=4\}$ is not in S.

Def 4.6: A collection of all the points $z\in\mathbb{C}$ that are on the distance $\varepsilon>0$ from z_0 is called ε-neighborhood of $z_0\in\mathbb{C}$

For $\varepsilon>0$, $z_0\in\mathbb{C}$: ε-neighborhood of z_0 is the set

$$\{z\in\mathbb{C} \mid |z-z_0|<\varepsilon\}$$

Having all these definitions, now we can study the continuity of a complex function.

Continuity

Def 5.1: The function $f(z)$ is continuous at the point z_0 if it is defined at a neighborhood of z_0 (and at z_0) and $\lim_{z\to z_0} f(z) = f(z_0)$

Namely: f is defined at z_0, at some ε-neighborhood of z_0, the limit above exists and it is equal to the value of the function at the point z_0.

The function $f(z)$ is continuous in some domain S if it is continuous at every point $z\in S$

As a consequence of the limit laws in Prop 3.1, we have the following (similar) proposition about the continuity.

Prop 5.1: If $f(z)$ and $g(z)$ are continuous at $z_0\in\mathbb{C}$, then so are $f+g$ and $f\cdot g$, and if $g(z_0)\neq 0$, then $\frac{f}{g}$ is continuous at z_0.

We also have an analogue of Prop 3.2:

Prop 5.2: The function $f(z)=u(x,y)+iv(x,y)$ is continuous at the point $z_0=x_0+iy_0 \iff$ the real functions $u(x,y)$, $v(x,y)$ are continuous at the point (x_0,y_0).

Let us see some examples of continuous functions.

Ex 7.1: a) It is obvious that the constant function is continuous everywhere: $f(z)=c$ is continuous at every $z\in\mathbb{C}$

b) It is also clear that the identity function is continuous everywhere: $f(z)=z$ is continuous at every $z\in\mathbb{C}$ (Check - from Def 5.1!)

c) Combining a), b), and Prop 5.1: we conclude that the linear function is continuous everywhere: For constants $a,b\in\mathbb{C}$ $f(z)=az+b$ is continuous at every $z\in\mathbb{C}$.

Ex 7.2: Prove that z^n, $n\in\mathbb{N}$ is continuous for all $z\in\mathbb{C}$.

Pf: We prove it by checking the definition Def 5.1 at some $z_0\in\mathbb{C}$. Consider:

$$z^n - z_0^n = (z-z_0)(z^{n-1}+z^{n-2}z_0+z^{n-3}z_0^2+\dots+z z_0^{n-2}+z_0^{n-1})$$

Denote $|z|=r$, $|z_0|=r_0$. Then, we obtain

$$|z^n-z_0^n| \le |z-z_0|(r^{n-1}+r^{n-2}r_0+\dots+r r_0^{n-2}+r_0^{n-1})$$

Let us look at all the z-s inside the disc of radius δ centered at z_0: $\{z\in\mathbb{C} \mid |z-z_0|<\delta\}$. For every z in this set

$r = |z| = |z - z_0 + z_0| \le |z - z_0| + |z_0| < \delta + r_0$, thus

$$|z^n - z_0^n| \le |z - z_0| \left((r_0+\delta)^{n-1} + (r_0+\delta)^{n-2} r_0 + \dots + (r_0+\delta) r_0^{n-2} + r_0^{n-1} \right)$$
$$\le \delta n (r_0+\delta)^{n-1}$$

The choice of δ small enough so that $|z^n - z_0^n|$ is smaller than the given ε finishes the proof. ∎

Combining Prop 5.1, Ex 7.2, and Ex 7.1 a), b), we conclude:

Prop 5.3: Every polynomial $P_n(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$ with complex coefficients $a_0, a_1, a_2, a_3, \dots, a_n \in \mathbb{C}$ is continuous at every $z \in \mathbb{C}$. Furthermore, every rational function $f(z) = \frac{P_n(z)}{Q_m(z)}$ (P_n, Q_m polynomials of degree n, m respectively) is continuous at every $z \in \mathbb{C}$, except possibly at the roots of $Q_m(z)$.

"Possibly" since if there is a common root to $P_n(z)$ and $Q_m(z)$ that can be factorized, the function may be continuous at that root.

We said that rational functions are continuous except possibly at the roots of the denominator, however, recalling Def 3.3:

Def 3.3 $\Rightarrow$ Rational functions can be regarded as continuous everywhere on $\mathbb{C} \cup \{\infty\}$.

In practice, one of the best ways of dealing with "∞" in limits is to use substitution $z = \frac{1}{\zeta}$ and let $\zeta \to 0$

Ex 7.3: 1) $\lim\limits_{z \to -1} \frac{iz+3}{z+1} = \infty$ since $\lim\limits_{z\to -1} \frac{z+1}{iz+3} = \frac{0}{-i+3} = 0$

2) $\lim\limits_{z\to\infty} \frac{1}{z+2i} = \lim\limits_{\zeta\to 0} \frac{1}{\frac{1}{\zeta}+2i} = \lim\limits_{\zeta\to 0} \frac{\zeta}{1+2i\zeta} = \lim\limits_{\zeta\to 0} \zeta \lim\limits_{\zeta\to 0} \frac{1}{2i\zeta+1} = 0 \cdot 1 = 0$

3) $\lim\limits_{z\to\infty} \frac{5z+i}{z+i} = \lim\limits_{\zeta\to 0} \frac{\frac{5}{\zeta}+i}{\frac{1}{\zeta}+i} = \lim\limits_{\zeta\to 0} \frac{5+i\zeta}{1+i\zeta} = \lim\limits_{\zeta\to 0}(5+i\zeta) \lim\limits_{\zeta\to 0} \frac{1}{1+i\zeta} = 5\cdot 1 = 5$

Ex 7.4: Is $\frac{z^3+8i}{z+1}$ continuous at $z = 2i$?

Sol: Since $2i+1 \ne 0$, by Prop 5.1:

$\lim\limits_{z\to 2i} \frac{z^3+8i}{z+1} = \lim\limits_{z\to 2i}(z^3+8i) \lim\limits_{z\to 2i} \frac{1}{z+1} = (8i^3+8i)\frac{1}{2i+1} = \frac{-8i+8i}{2i+1} = 0$

$\Rightarrow$ continuous: the function is defined at $z = 2i$ and in a neighborhood of $z = 2i$, the limit exists and equal to the value of the function $\Rightarrow$ the function is continuous at $z = 2i$.

Lectures 8+9.

We have studied the limits and the continuity of complex functions. Now we will study what does it mean for the complex function to be differentiable.

Differentiability

Def 6.1 Let $f(z)$ be defined in a neighborhood of a point $z_0 \in \mathbb{C}$. If the limit $\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists, then f is differentiable at z_0 and

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}.$$

Alternative notation: Denote $\Delta z = z - z_0$. Then

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$$

Ex 8.1: Using Def 6.1 compute the derivative of

a) $f(z) = z$ b) $f(z) = z^2$

Sol: a) $\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{z_0+\Delta z - z_0}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta z}{\Delta z} = 1$

$\Rightarrow f'(z)$ exists for all z and $f'(z) = 1$

b) $\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z_0+\Delta z)^2 - z_0^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{z_0^2 + 2z_0\Delta z + \Delta z^2 - z_0^2}{\Delta z}$

$= \lim_{\Delta z \to 0} \left(2z_0 \frac{\Delta z}{\Delta z} + \frac{\Delta z^2}{\Delta z}\right) = 2z_0 + \lim_{\Delta z \to 0} \Delta z = 2z_0$

$\Rightarrow f'(z)$ exists for all z and $f'(z) = 2z$

Ex 8.2: Prove that for any $n \in \mathbb{N}$ $(z^n)' = nz^{n-1}$ for all $z \in \mathbb{C}$.

Pf: Recall: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$, where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

$\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z_0+\Delta z)^n - z_0^n}{\Delta z} =$

$= \lim_{\Delta z \to 0} \frac{z_0^n + nz_0^{n-1}\Delta z + \frac{n(n-1)}{2} z_0^{n-2}\Delta z^2 + \ldots + \Delta z^n - z_0^n}{\Delta z}$

$= \lim_{\Delta z \to 0} \left(nz_0^{n-1} + \frac{n(n-1)}{2} z_0^{n-2}\Delta z + \ldots + \Delta z^{n-1}\right) = nz_0^{n-1}$

$\Rightarrow f'(z)$ exists for all z and $f'(z) = nz^{n-1}$ ☑

These examples suggest that the differentiation of complex functions is similar to the differentiation of real functions. Let us see examples that it may be very different!

Ex 8.3 Compute the derivative of $f(z) = \bar{z}$

Sol: $\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - \bar{z}_0}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{(z_0+\Delta z)} - \bar{z}_0}{\Delta z} = \lim_{\Delta z \to 0} \frac{\bar{z}_0 + \overline{\Delta z} - \bar{z}_0}{\Delta z} =$

$= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$ – doesn't exist! (Ex 5.2)

We have proved (Ex 5.2) that this limit does not exist by looking at the limits along real and imaginary axes and showing that they do not agree.

$\Rightarrow f(z) = \bar{z}$ is not differentiable at any $z \in \mathbb{C}$.

Ex 8.4: Compute the derivative of $f(z) = |z|^2$

Sol: $\lim_{\Delta z \to 0} \frac{|z+\Delta z|^2 - |z|^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z+\Delta z)\overline{(z+\Delta z)} - |z|^2}{\Delta z}$

$= \lim_{\Delta z \to 0} \frac{z\bar{z} + \Delta z \bar{z} + z\overline{\Delta z} + \Delta z \overline{\Delta z} - z\bar{z}}{\Delta z} = \lim_{\Delta z \to 0} \left(\bar{z} + \overline{\Delta z} + z \frac{\overline{\Delta z}}{\Delta z}\right)$ (*)

If $z \neq 0$, then $\lim_{\Delta z \to 0} z \frac{\overline{\Delta z}}{\Delta z}$ does not exist! (tends to z along the real axis and to $(-z)$ along imaginary axis $\Rightarrow$ does not exist)

$\Rightarrow$ The limit (*) does not exist $\Rightarrow f'(z)$ does not exist

If $z = 0$: $\lim_{\Delta z \to 0} z \cdot \frac{\overline{\Delta z}}{\Delta z} = 0$, $\bar{z} = 0 \Rightarrow$ the limit (*) exists and $f'(0) = 0$.

Thus, $f(z) = |z|^2$ is differentiable only at $z = 0$.

Now we observe how we can construct complicated differentiable functions from elementary ones. The following proposition is analogous to the real case.

Prop 6.1: If $c \in \mathbb{C}$ is a constant and $f(z), g(z)$ are differentiable at z, then:

1) $c' = 0$
2) $(cf(z))' = cf'(z)$
3) $f+g$ is differentiable at z and $(f+g)'(z) = f'(z) + g'(z)$
4) $f \cdot g$ is differentiable at z and $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$
5) If $g(z) \neq 0$, then $\frac{f}{g}$ is differentiable at z and

$$\left(\frac{f}{g}\right)'(z) = \frac{g(z)f'(z) - f(z)g'(z)}{(g(z))^2}$$

6) Chain rule: If $f(z)$ is differentiable at z_0 and $g(z)$ is differentiable at $w_0 = f(z_0)$, then the composed function $(g \circ f)(z) = g(f(z))$ is differentiable at z_0 and

$$(g \circ f)'(z_0) = g'(w_0) f'(z_0) = g'(f(z_0)) f'(z_0)$$

Pf: EXERCISE (The details follow from Def 6.1, Prop 3.1; the chain rule – proof follows from Def 6.1 as in the real case)

We have seen that the constant function, the identity function and z^n are differentiable everywhere. Since any polynomial can be constructed out of these functions, using Prop 6.1 we conclude:

Combining Ex 8.1, Ex 8.2, and Prop 6.1 1), 3), we obtain:

Cor 6.1: Any polynomial $P_n(z) = a_0 + a_1 z + \dots + a_n z^n$, $a_0, \dots, a_n \in \mathbb{C}$ is differentiable everywhere in $\mathbb{C}$ and

$$(P_n)'(z) = a_1 + 2a_2 z + 3a_3 z^2 + \dots + n a_n z^{n-1}$$

Prop 6.1 with Cor 6.1 imply

Cor 6.2: Any rational function $f(z) = \frac{P_n(z)}{Q_m(z)}$ (P_n, Q_m polynomials of degree n, m respectively) is differentiable everywhere in $\mathbb{C}$, except possibly at the roots of $Q_m(z)$, and

$$f'(z) = \frac{P_n'(z)Q_m(z) - P_n(z)Q_m'(z)}{(Q_m(z))^2} \quad (Q_m(z) \neq 0)$$

Ex 8.5: Compute the derivative of $f(z) = \frac{z^3 - 5z + 2i}{z^2 - 3iz}$

Sol: Combining Prop 6.4, Cor 6.2, and Ex 8.2 we obtain

$$f'(z) = \frac{(3z^2-5)(z^2-3iz) - (z^3-5z+2i)(2z-3i)}{(z^2-3iz)^2}$$

$$= \frac{z^4 - 6iz^3 + 5z^2 - 4iz - 6}{(z^2-3iz)^2}$$

(exists for $z \neq 0, 3i$, for which $z^2 - 3iz = 0$ and the numerator is not 0!)

Next proposition provides a relation between the continuity and differentiability.

Prop 6.5: If f is differentiable at z_0, then f is continuous at z_0.

Pf: Observe that

$$\lim_{z\to z_0} (f(z) - f(z_0)) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}(z-z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} \lim_{z\to z_0}(z-z_0)$$
$$= f'(z_0)\cdot 0 = 0$$

$$\Rightarrow \lim_{z\to z_0} (f(z) - f(z_0)) = \left(\lim_{z\to z_0} f(z)\right) - f(z_0) = 0 \Leftrightarrow \lim_{z\to z_0} f(z) = f(z_0) \quad \blacksquare$$

Rmk: The converse is false! $f(z) = \bar{z}$ is continuous everywhere, yet differentiable nowhere!

This leads to the following natural question:

Q-n: What are the necessary and sufficient conditions for a complex function to be differentiable?

We are going to state and partially prove these conditions.

Recall: A real function $u(x,y)$ is differentiable at the point (x_0,y_0) if both partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ exist at (x_0,y_0), and

$$\lim_{(x,y)\to(x_0,y_0)} \frac{u(x,y)-u(x_0,y_0)-\frac{\partial u}{\partial x}(x_0,y_0)(x-x_0)-\frac{\partial u}{\partial y}(x_0,y_0)(y-y_0)}{|(x-x_0)^2+|y-y_0|^2|^{1/2}} = 0$$

Namely, the function $\left(u(x_0,y_0)+\frac{\partial u}{\partial x}(x_0,y_0)(x-x_0)+\frac{\partial u}{\partial y}(x_0,y_0)(y-y_0)\right)$ is a good approximation to $u(x,y)$ near the point (x_0,y_0).

In particular, it means that:

There exists a neighborhood of (x_0,y_0) s.t. $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ are defined everywhere in this neighborhood and continuous at (x_0,y_0)

Now we are ready to state the conditions

Thm 6.1: The function $f(z)=u(x,y)+iv(x,y)$ is differentiable at $z_0=x_0+iy_0$ if and only if $u(x,y)$ and $v(x,y)$ are differentiable at (x_0,y_0) and the following conditions, called Cauchy-Riemann equations, hold:

$$\begin{cases} \frac{\partial u}{\partial x}(x_0,y_0)=\frac{\partial v}{\partial y}(x_0,y_0) \\ \frac{\partial u}{\partial y}(x_0,y_0)=-\frac{\partial v}{\partial x}(x_0,y_0) \end{cases} \quad \text{CR equations}$$

Furthermore: $f'(z_0)=\frac{\partial u}{\partial x}(x_0,y_0)+i\frac{\partial v}{\partial x}(x_0,y_0)$

The contrapositive statement: if Cauchy-Riemann equations fail, then the function is not differentiable.

However, it is not true that if CR equations hold, then the function is differentiable, namely, we cannot drop the assumption that u and v are differentiable at (x_0,y_0)

Ex 8.6 Let $z=x+iy$, $f(z)=\sqrt{|xy|}$, namely: $u(x,y)=\sqrt{|xy|}$, $v(x,y)=0$

CHECK: $f(z)$ satisfies CR equations at $z=0$:

$$\frac{\partial u}{\partial x}(0,0)=\frac{\partial v}{\partial y}(0,0)=\frac{\partial u}{\partial y}(0,0)=\frac{\partial v}{\partial x}(0,0)=0$$

CHECK: $u(x,y)$ is not differentiable at $(0,0)$.

Claim: $f'(0)$ does not exist

Pf: Need to show that $\lim_{z\to 0}\frac{f(z)-f(0)}{z-0}=\lim_{z\to 0}\frac{f(z)}{z}$ does not exist.

We do that by showing that on different paths the limit takes different values.

Let $x=\alpha r$, $y=\beta r$, where $\alpha,\beta \in \mathbb{R}$, $\alpha,\beta \neq 0$ constants. Assume $r>0$ ($r<0$ in the same way)

$$\lim_{z\to 0}\frac{f(z)}{z} = \lim_{(x,y)\to(0,0)}\frac{\sqrt{|xy|}}{x+iy} = \lim_{r\to 0}\frac{r\sqrt{|\alpha\beta|}}{r(\alpha+i\beta)} = \lim_{r\to 0}\frac{\sqrt{|\alpha\beta|}}{\alpha+i\beta} = \frac{\sqrt{|\alpha\beta|}}{\alpha+i\beta}$$

Namely, for different α and β the limit takes different values, thus does not exist.

This example shows that it is not enough for the sufficient part that the partial derivatives are just defined at a point, they need to be defined in a neighborhood and to be continuous at the point. For the necessary part it is enough though, and we will prove this direction now.

First, we state a corollary that gives polar form of Cauchy-Riemann equations.

Cor 6.3 [Polar form of Cauchy-Riemann equations]

Let $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$, where $r=|z|$, $\theta = \arg z \bmod 2\pi$.

Let $f(z) = \psi(r,\theta) + i\varphi(r,\theta)$. Then

$$\frac{\partial\psi}{\partial r} = \frac{1}{r}\frac{\partial\varphi}{\partial\theta} \quad \text{and} \quad \frac{\partial\psi}{\partial\theta} = -r\frac{\partial\varphi}{\partial r}$$

Moreover: $f'(z) = e^{-i\theta}\left(\frac{\partial\psi}{\partial r} + i\frac{\partial\varphi}{\partial r}\right)$

Now we prove the following form of the necessary condition:

Prop 6.6: If $f(z) = u(x,y) + iv(x,y)$ is differentiable at $z_0 = x_0 + iy_0$, then $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ all exist at (x_0,y_0), and

$$\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0) \quad \text{and} \quad \frac{\partial u}{\partial y}(x_0,y_0) = -\frac{\partial v}{\partial x}(x_0,y_0)$$

This statement is weaker as we are not proving that u and v are differentiable at (x_0,y_0).

Pf: Since f is differentiable at z_0, the following limit exists:

$$f'(z_0) = \lim_{\Delta z\to 0}\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} =$$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{u(x_0+\Delta x, y_0+\Delta y) + iv(x_0+\Delta x, y_0+\Delta y) - (u(x_0,y_0) + iv(x_0,y_0))}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\left[\frac{u(x_0+\Delta x, y_0+\Delta y) - u(x_0,y_0)}{\Delta x + i\Delta y} + i\,\frac{v(x_0+\Delta x, y_0+\Delta y) - v(x_0,y_0)}{\Delta x + i\Delta y}\right] \quad (*)$$

Since the limit ⊛ exists, it is unique and independent of path $\Delta z \to 0$. We compute this limit along 2 different paths:

1) $\Delta x \to 0, \Delta y = 0$ 2) $\Delta x = 0, \Delta y \to 0$

1) $$f'(z_0) = \lim_{\substack{\Delta x \to 0 \\ \Delta y = 0}} \left[\frac{u(x_0+\Delta x, y_0) - u(x_0,y_0)}{\Delta x} + i \frac{v(x_0+\Delta x, y_0) - v(x_0,y_0)}{\Delta x} \right]$$
$$= \frac{\partial u}{\partial x}(x_0,y_0) + i \frac{\partial v}{\partial x}(x_0,y_0)$$

2) $$f'(z_0) = \lim_{\substack{\Delta x = 0 \\ \Delta y \to 0}} \left[\frac{u(x_0, y_0+\Delta y) - u(x_0,y_0)}{i\Delta y} + i \frac{v(x_0, y_0+\Delta y) - v(x_0,y_0)}{i\Delta y} \right]$$
$$= \frac{1}{i}\frac{\partial u}{\partial y}(x_0,y_0) + \frac{\partial v}{\partial y}(x_0,y_0) = -i\frac{\partial u}{\partial y}(x_0,y_0) + \frac{\partial v}{\partial y}(x_0,y_0)$$

Since on both paths we get the same limit, we conclude that
$$\frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0) = -i\frac{\partial u}{\partial y}(x_0,y_0) + \frac{\partial v}{\partial y}(x_0,y_0)$$

Comparing the real and imaginary parts we obtain
$$\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0) \quad \text{and} \quad -\frac{\partial u}{\partial y}(x_0,y_0) = \frac{\partial v}{\partial x}(x_0,y_0) \quad \blacksquare$$

Ex 8.7 Where $f(x,y) = x^3 - 3xy^2 + i(3x^2 - y^3)$ is differentiable?

Sol: Let us check where $u = x^3 - 3xy^2$, $v = 3x^2 - y^3$ are differentiable and CR equations hold

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2$ $\qquad \frac{\partial v}{\partial x} = 6x$

$\frac{\partial u}{\partial y} = -6xy$ $\qquad \frac{\partial v}{\partial y} = -3y^2$

$\Rightarrow u, v$ are differentiable at every (x,y). CR equations:

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \iff \begin{cases} 3x^2 - 3y^2 = -3y^2 \to \text{holds only for } x=0 \\ -6xy = 6x \to \text{since } x=0, \text{ holds for any } y \end{cases}$$

$\Rightarrow f(z)$ is differentiable at all points of the form $(0,y)$

and $f'(0,y) = \frac{\partial u}{\partial x}(0,y) + i\frac{\partial v}{\partial x}(0,y) = -3y^2$.