Problem Set 2 – Solutions.

1) Euler's formula: for any $\theta \in \mathbb{R}$ $e^{i\theta} = \cos\theta + i\sin\theta$

a) We have defined $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$, $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$

Thus, we obtain

$$\sin^2 z + \cos^2 z = \left(\tfrac{1}{2i}(e^{iz} - e^{-iz})\right)^2 + \left(\tfrac{1}{2}(e^{iz} + e^{-iz})\right)^2 =$$
$$= -\tfrac{1}{4}(e^{2iz} + e^{-2iz} - 2e^{iz}e^{-iz}) + \tfrac{1}{4}(e^{2iz} + e^{-2iz} + 2e^{iz}e^{-iz})$$
$$= \tfrac{1}{4}(e^{2iz} + e^{-2iz} + 2e^{iz}e^{-iz} - e^{2iz} - e^{-2iz} + 2e^{iz}e^{-iz})$$
$$= \tfrac{1}{4}4e^{iz}e^{-iz} = e^{iz-iz} = e^0 = 1.$$

b) Let us look at the right hand side.

$$\cos z_1 \cos z_2 - \sin z_1 \sin z_2 = \tfrac{1}{2}(e^{iz_1} + e^{-iz_1})\tfrac{1}{2}(e^{iz_2} + e^{-iz_2})$$
$$- \tfrac{1}{2i}(e^{iz_1} - e^{-iz_1})\tfrac{1}{2i}(e^{iz_2} - e^{-iz_2})$$
$$= \tfrac{1}{4}(e^{iz_1}e^{iz_2} + e^{-iz_1}e^{iz_2} + e^{iz_1}e^{-iz_2} + e^{-iz_1}e^{-iz_2}) +$$
$$\underset{i^2=-1!}{\uparrow} \quad + \tfrac{1}{4}(e^{iz_1}e^{iz_2} - e^{iz_1}e^{-iz_2} - e^{-iz_1}e^{iz_2} + e^{-iz_1}e^{-iz_2})$$
$$= \tfrac{1}{4}\Big(2e^{iz_1}e^{iz_2} + e^{-iz_1}e^{iz_2} - e^{iz_1}e^{-iz_2} + e^{iz_1}e^{-iz_2} - e^{-iz_1}e^{iz_2} +$$
$$+ 2e^{-iz_1}e^{-iz_2}\Big) = \tfrac{1}{4}\left(2e^{iz_1}e^{iz_2} + 2e^{-iz_1}e^{-iz_2}\right)$$
$$= \tfrac{1}{2}(e^{iz_1}e^{iz_2} + e^{-iz_1}e^{-iz_2}) = \tfrac{1}{2}\left(e^{i(z_1+z_2)} + e^{-i(z_1+z_2)}\right)$$
$$\underset{\text{Def!}}{=} \cos(z_1 + z_2)$$

c) $\boxed{\Leftarrow}$ If $z = \frac{\pi}{2} + \pi k \Rightarrow \cos(\frac{\pi}{2} + \pi k) = 0$

$\boxed{\Rightarrow}$ Assume $\cos z = 0$. Then

$$\cos z = \tfrac{1}{2}(e^{iz} + e^{-iz}) = \tfrac{1}{2}\left(\frac{e^{2iz}+1}{e^{iz}}\right) = 0$$

$\Leftrightarrow e^{2iz} + 1 = 0$. Let $z = x + iy$, then by Euler's formula, we get

$$e^{2iz} + 1 = e^{2i(x+iy)} + 1 = e^{2ix-2y} + 1 = e^{2ix}e^{-2y} + 1 = 0$$

$\Leftrightarrow \quad e^{-2y}(\cos 2x + i\sin 2x) = -1 = -1 + i\cdot 0$

Now we compare real and imaginary parts:

$$\begin{cases} e^{-2y}\cos 2x = -1 \\ e^{-2y}\sin 2x = 0 \end{cases} \Rightarrow y = 0 \Rightarrow (*) \begin{cases} \cos 2x = -1 \\ \sin 2x = 0 \end{cases}$$

$\cos 2x = -1 \Leftrightarrow 2x = \pi + 2\pi k = \pi(2k+1) \Rightarrow x = \frac{\pi}{2} + \pi k$, $k \in \mathbb{Z}$

$\sin 2x = 0 \Leftrightarrow 2x = \pi k$, $k \in \mathbb{Z}$

Thus, the solution of the system $(*)$ is $x = \frac{\pi}{2} + \pi k$.

2) a) Let $z = x+iy$. By Euler's formula

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$$

Thus: $e^z - i = 0 \iff e^x \cos y + i(e^x \sin y - 1) = 0$

We get the following system of equations:

$$\begin{cases} e^x \cos y = 0 \\ e^x \sin y - 1 = 0 \end{cases} \iff \begin{cases} e^x \cos y = 0 \\ e^x \sin y = 1 \end{cases}$$

Since $e^x \neq 0$ for all $x \in \mathbb{R}$, we get from the first equation that $\cos y = 0$, namely $y = \frac{\pi}{2} + \pi k$, $k \in \mathbb{Z}$. Now we plug this y into the second equation and get

$$e^x \sin\left(\frac{\pi}{2} + \pi k\right) = 1$$

Since $\sin\left(\frac{\pi}{2} + \pi k\right) = (-1)^k$ we obtain that for <u>even</u> k
$k = 0, \pm 2, \pm 4, \dots$ $\sin\left(\frac{\pi}{2} + \pi k\right) = 1$.

$$\Rightarrow e^x (-1)^{2m} = e^x = 1.$$

Since $e^x > 1$ for all $x > 0$ and $e^x \leq 1$ for all $x \leq 0$ we conclude that $x = 0$. Thus,

$$z = 0 + i\left(\frac{\pi}{2} + 2\pi m\right) = i\left(\frac{\pi}{2} + 2\pi m\right)$$

b) As before: $z = x+iy$, then

$$e^z = e^{x+iy} = 1 \iff e^x(\cos y + i\sin y) = 1$$

$$\iff \begin{cases} e^x \cos y = 1 \\ e^x \sin y = 0 \iff \sin y = 0 \iff y = \pi k, \ k \in \mathbb{Z} \end{cases}$$

$\Rightarrow \cos \pi k = (-1)^k \Rightarrow k$ should be even, namely $k = 2m$, $m \in \mathbb{Z}$. As before we conclude that $x = 0$ and get

$$e^z = 1 \iff z = 0 + i\,2\pi m = i\,2\pi m$$

c) Again $z = x+iy$

$$e^{iz} + 1 - i = e^{i(x+iy)} + 1 - i = e^{ix-y} + 1 - i = e^{-y} e^{ix} + 1 - i = 0$$

$$\iff e^{-y}(\cos x + i\sin x) = -1 + i$$

$$\iff \begin{cases} e^{-y} \cos x = -1 \\ e^{-y} \sin x = 1 \end{cases} \iff \begin{cases} \cos x = -e^y \\ \sin x = e^y \end{cases} \iff \cos x = -\sin x$$

$\iff x = \frac{3\pi}{4} + \pi k$, $k \in \mathbb{Z}$. Thus, we get

$$\cos\left(\frac{3\pi}{4} + \pi k\right) = \frac{(-1)^{2k+1}}{\sqrt{2}} = -e^y \iff \frac{(-1)^{2k+2}}{\sqrt{2}} = e^y \iff e^y = \frac{1}{\sqrt{2}}$$

$$\iff y = \ln \frac{1}{\sqrt{2}} = -\ln \sqrt{2}$$

Thus, we have : $x = \frac{3\pi}{4} + \pi k$, $k \in \mathbb{Z}$, $y = -\ln\sqrt{2}$ are solutions (and there are no more solutions).

3) a) 1) $f(z) = \frac{z-1}{z+1}$

Since $\overline{z+1} = \bar{z} + \bar{1} = \bar{z} + 1$ we have

$$\frac{z-1}{z+1} = \frac{z-1}{z+1} \cdot \frac{\bar{z}+1}{\bar{z}+1} \underset{\downarrow}{=} \frac{z\bar{z} - \bar{z} + z - 1}{|z+1|^2} = \frac{|z|^2 - 1 + (z-\bar{z})}{|z+1|^2}$$

$$(z+1)(\bar{z}+1) = |z+1|^2$$

$$\underset{\downarrow}{=} \frac{|z|^2 - 1 + 2i\operatorname{Im} z}{|z+1|^2}$$

Recall: $\frac{1}{2i}(z - \bar{z}) = \operatorname{Im} z$

Note : $|z|^2, |z+1|^2, 2\operatorname{Im} z \in \mathbb{R}$, thus we get

$$\frac{z-1}{z+1} = \frac{|z|^2-1}{|z+1|^2} + i\frac{2\operatorname{Im} z}{|z+1|^2}, \text{ namely}$$

$$\operatorname{Re}\frac{z-1}{z+1} = \frac{|z|^2-1}{|z+1|^2} \qquad \operatorname{Im}\frac{z-1}{z+1} = +\frac{2\operatorname{Im} z}{|z+1|^2}$$

Alternatively: Let $z = x+iy$. Then

$$\frac{z-1}{z+1} = \frac{x+iy-1}{x+iy+1} = \frac{(x-1)+iy}{(x+1)+iy} \underset{\downarrow}{=} \frac{(x-1)+iy}{(x+1)+iy} \cdot \frac{(x+1)-iy}{(x+1)-iy} =$$

$$\overline{(x+1)+iy} = (x+1) - iy$$

$$= \frac{(x-1)(x+1) + y^2 + iy(x+1) - iy(x-1)}{(x+1)^2 + y^2} = \frac{x^2 - 1 + y^2 + 2iy}{(x+1)^2 + y^2} =$$

$$= \frac{x^2+y^2-1}{(x+1)^2+y^2} + i\frac{2y}{(x+1)^2+y^2} \Rightarrow \operatorname{Re}\frac{z-1}{z+1} = \frac{x^2+y^2-1}{(x+1)^2+y^2}$$

$$\operatorname{Im}\frac{z-1}{z+1} = \frac{2y}{(x+1)^2+y^2}$$

2) $f(z) = 3z^2 + \frac{1}{z}$: let $z = x+iy$. Then

$$3(x+iy)^2 + \frac{1}{x+iy} = 3(x^2 - y^2 + 2ixy) + \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} =$$

$$= 3(x^2 - y^2 + 2ixy) + \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} = \left(3x^2 - 3y^2 + \frac{x}{x^2+y^2}\right) +$$

$$+ i\left(6xy - \frac{y}{x^2+y^2}\right) \Rightarrow \operatorname{Re} f(z) = 3x^2 - 3y^2 + \frac{x}{x^2+y^2}$$

$$\operatorname{Im} f(z) = 6xy - \frac{y}{x^2+y^2}$$

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b) 1) $f(x,y) = x+2y+i(3y-x)$

$x = \frac{1}{2}(z+\bar{z}) \quad y = \frac{1}{2i}(z-\bar{z})$

$\Rightarrow x+2y+i(3y-x) = \frac{1}{2}(z+\bar{z}) + \frac{2}{2i}(z-\bar{z}) + i\left(\frac{3}{2i}(z-\bar{z}) - \right.$

$\left. - \frac{1}{2}(z+\bar{z})\right) = \frac{1}{2}(z+\bar{z}) - i(z-\bar{z}) + \frac{3}{2}(z-\bar{z}) - \frac{i}{2}(z-\bar{z})$

$\frac{1}{i} = -i$

$= \frac{1}{2}(z+\bar{z}+3z-3\bar{z}) - i(z-\bar{z}+\frac{1}{2}z-\frac{1}{2}\bar{z})$

$= \frac{1}{2}(4z-2\bar{z}) - i(\frac{3}{2}z - \frac{3}{2}\bar{z}) = 2z-\bar{z} - \frac{3}{2}iz + \frac{3}{2}i\bar{z}$

$= z(2-i\frac{3}{2}) - \bar{z}(1-i\frac{3}{2})$

2) $f(x,y) = x^2+y^2-i(4x-y)$

$x^2+y^2-i(4x-y) = \frac{1}{4}(z+\bar{z})^2 + \left(\frac{1}{2i}(z-\bar{z})\right)^2 - i\left(\frac{4}{2}(z+\bar{z}) - \frac{1}{2i}(z-\bar{z})\right)$

$= \frac{1}{4}(z^2+\bar{z}^2+2z\bar{z}) - \frac{1}{4}(z^2+\bar{z}^2-2z\bar{z}) - 2i(z+\bar{z}) + \frac{1}{2}(z-\bar{z})$

$= \frac{1}{4}(z^2+\bar{z}^2+2z\bar{z}-z^2-\bar{z}^2+2z\bar{z}) - z(2i-\frac{1}{2}) - \bar{z}(2i+\frac{1}{2})$

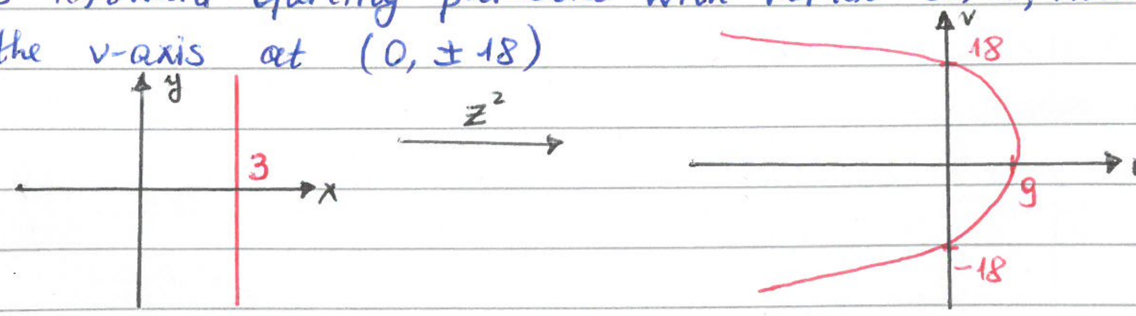
$= z\bar{z} - z(2i-\frac{1}{2}) - \bar{z}(2i+\frac{1}{2})$

4) a) 1) $w = z^2 : \quad (x,y) \longmapsto (\underbrace{x^2-y^2}_{u}, \underbrace{2xy}_{v})$. Set $x=3$, we get:

$\begin{cases} u = 9-y^2 \\ v = 6y \end{cases} \Rightarrow y = \frac{v}{6} \Rightarrow u = 9 - \frac{v^2}{36}$

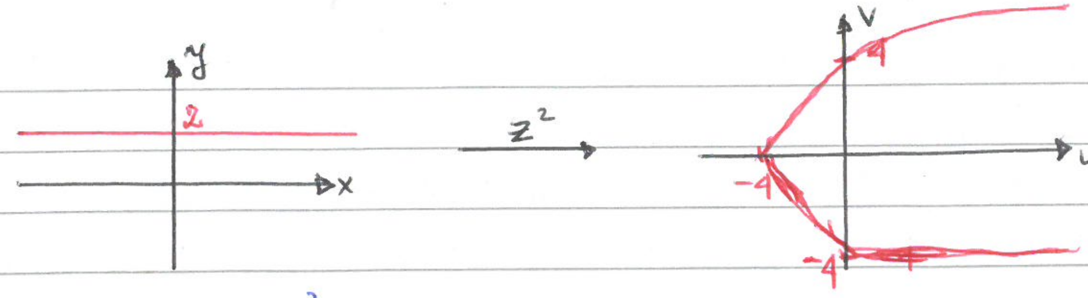
plug into first equation

This is leftward opening parabola with vertex $(9,0)$, intersecting the v-axis at $(0, \pm 18)$

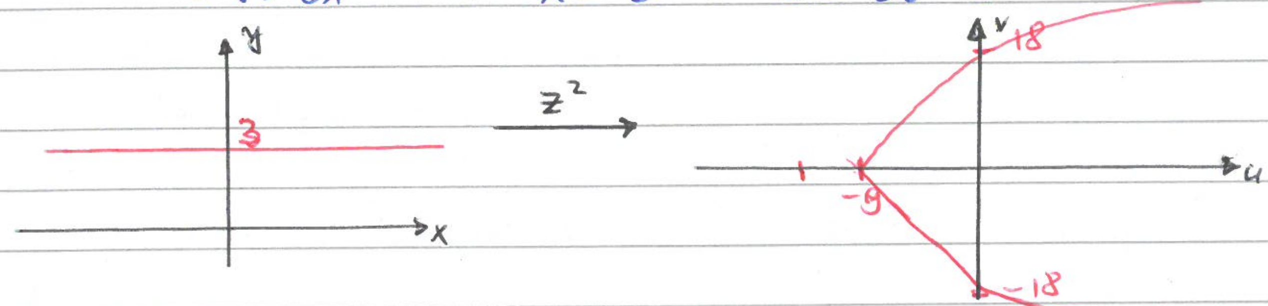


2) Let us find the images of the lines $y=2$ and $y=3$. The image of the domain $2<y<3$ will be between the image of $y=2$ and the image of $y=3$

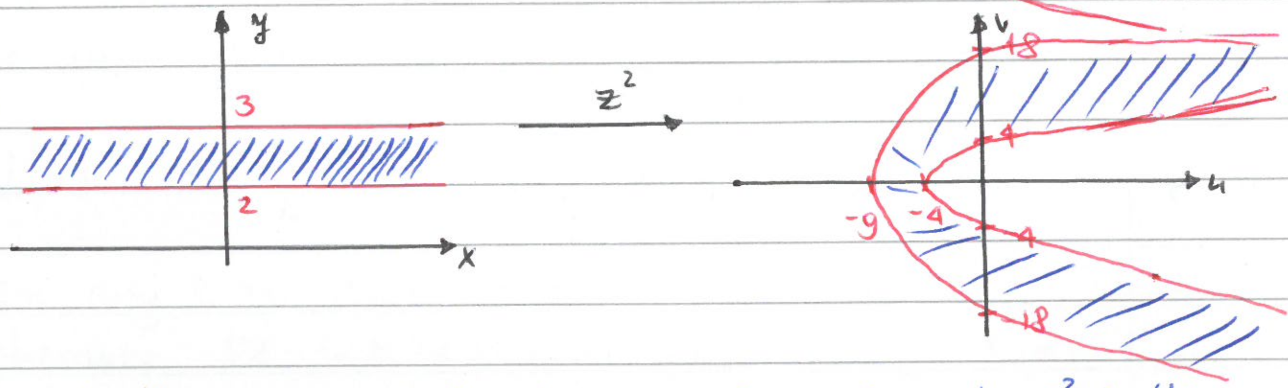
For $y=2$ we get the system: $\begin{cases} u = x^2-4 \\ v = 2x \end{cases} \Rightarrow x = \frac{v}{2}, \; u = \frac{v^2}{4} - 4$



$y=3$: $\begin{cases} u = x^2 - 9 \\ v = 6x \end{cases} \Rightarrow x = \frac{v}{6} \Rightarrow u = \frac{v^2}{36} - 9$



$\Rightarrow$



Namely, the image of the domain $2 < y < 3$ under z^2 is the domain between 2 parabolas $u = \frac{v^2}{4} - 4$ and $u = \frac{v^2}{36} - 9$

b) 1) $f(z) = \frac{1}{z}$: $z = x+iy$, $w = \frac{1}{z} = u+iv$

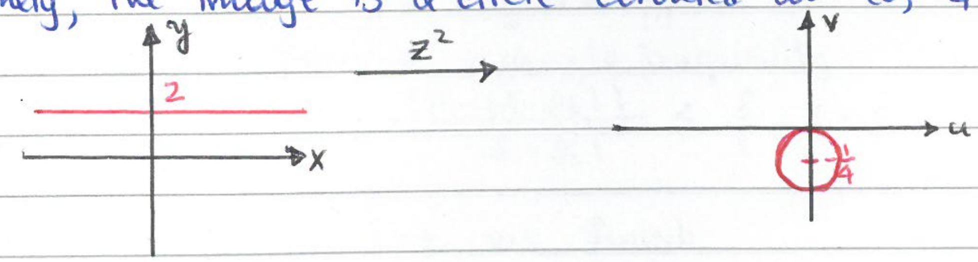
$\Rightarrow \frac{1}{x+iy} = u+iv \Leftrightarrow x+iy = \frac{1}{u+iv} = \frac{1}{u+iv} \cdot \frac{u-iv}{u-iv} = \frac{u^2}{u^2+v^2} - i\frac{v}{u^2+v^2}$

$\Rightarrow x = \frac{u^2}{u^2+v^2}$ $\quad y = -\frac{v}{u^2+v^2}$. Plug $y=2$ and get

$$2 = -\frac{v}{u^2+v^2} \Leftrightarrow 2u^2 + 2v^2 + v = 0 \Leftrightarrow u^2 + \left(v + \frac{1}{4}\right)^2 = \frac{1}{16}$$

complete the square in v

Namely, the image is a circle centered at $(0, -\frac{1}{4})$ of radius $\frac{1}{4}$



2) Represent the points in the interior of the unit circle in the first quadrant by $re^{i\theta}$ with $0 < \theta < \frac{\pi}{2}$

Note: this is a unit circle, thus $r < 1$.

Transformation $\frac{1}{z}$ maps $re^{i\theta}$ to $\frac{1}{r}e^{-i\theta} = \frac{1}{r}(\cos\theta - i\sin\theta)$
Since $r<1 \Rightarrow \frac{1}{r}>1$, therefore the points in the image are outside the unit circle. For $0<\theta<\frac{\pi}{2}$ $\cos\theta>0$, $\sin\theta>0$, hence under $\frac{1}{z}$ we get $\cos\theta>0$ and $-\sin\theta<0$, namely this is the fourth quadrant.

Alternatively: let $z=x+iy$. Describe the interior of the unit circle in the first quadrant as follows:

$$\{x,y\in\mathbb{R}:\ x,y\geq 0,\ \ x,y<1,\ \ x^2+y^2<1\}$$

Under $\frac{1}{z}$ we get:

$$(x,y)\longmapsto\left(\frac{x}{x^2+y^2},\ \frac{-y}{x^2+y^2}\right)$$

and again we see that these points lie in the exterior of the unit circle in the fourth quadrant (Why?)

5) a) Need to show: $\lim\limits_{z\to 1-2i}\frac{1}{z+2i}=1$. Namely:

for any $\varepsilon>0$ there exists $\delta>0$ (depends on ε!) such that whenever $|z-1+2i|<\delta$ we get $\left|\frac{1}{z+2i}-1\right|<\varepsilon$.
First: since z tends to $(1-2i)$ we can assume that $0<|z-1+2i|<1$. We have

$$\left|\frac{1}{z+2i}-1\right| = \left|\frac{1-z-2i}{z+2i}\right| = \left|\frac{z-(1-2i)}{z+2i}\right| = \frac{|z-(1-2i)|}{|z+2i|}$$

To bound the last expression from above we need an upper bound for the numerator and lower bound for the denominator. The numerator is $<\delta$ (which we will choose soon). The denominator:

$$|z+2i| = |z+1-1+2i| \underset{\substack{\downarrow\\ \text{reverse triangle inequality}}}{\geq} \big||z-1+2i|-1\big| > |-1| = 1$$

$$\Rightarrow \frac{1}{|z+2i|}<1 \Rightarrow \frac{|z-(1-2i)|}{|z+2i|} < \frac{\delta}{1} = \delta \leq \varepsilon$$

Thus, if we choose $\delta=\varepsilon$, we finish.

b) Write $z=r(\cos\theta+i\sin\theta)=r\cos\theta+ir\sin\theta$
Then $|z|^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$ and $\operatorname{Im} z = r\sin\theta$, thus $(\operatorname{Im} z)^2 = r^2\sin^2\theta$. If $z\to 0 \Rightarrow r\to 0$ (for any θ). Thus:

$$\lim_{z\to 0} \frac{(\operatorname{Im} z)^2}{|z|^2} = \lim_{r\to 0} \frac{r^2 \sin^2\theta}{r^2} = \lim_{r\to 0} \sin^2\theta = \sin^2\theta$$

For different θ the limit takes different values, thus the limit does not exist (If the limit exists it must be unique).

c) We start as in b) and get

$$\lim_{z\to 0} \frac{(\operatorname{Re} z)^2}{|z|} = \lim_{r\to 0} \frac{r^2\cos^2\theta}{r} = \lim_{r\to 0} r\cos^2\theta = 0 \text{ for any } \theta!$$

Thus, we have the same limit on every path => the limit exists and it is equal to 0.

Alternatively: let $z = x+iy$. We need to show that for any $\varepsilon > 0$ there exists $\delta > 0$ such that if

$|z-0| = |x+iy| = \sqrt{x^2+y^2} < \delta$, then $\left| \frac{x^2}{\sqrt{x^2+y^2}} - 0 \right| < \varepsilon$

Since $y^2 \geq 0$ always we have $\sqrt{x^2+y^2} \geq \sqrt{x^2} = |x|$. Therefore

$$\left| \frac{x^2}{\sqrt{x^2+y^2}} \right| \leq \left| \frac{x^2}{x} \right| = |x| < \delta \quad (\text{since } \sqrt{x^2+y^2} < \delta)$$

Choose $\delta = \varepsilon$ and finish. ▨

d) $\displaystyle\lim_{z\to 3i} \frac{z^2-4i}{2z^2+2i}$

Since for $z = 3i$ the value of the denominator

$$2(3i)^2 + 2i = -18+2i \neq 0$$

we can use Prop 3.1 3) and get

$$\lim_{z\to 3i} \frac{z^2-4i}{2z^2+2i} = \lim_{z\to 3i} (z^2-4i) \lim_{z\to 3i} \frac{1}{2z^2+2i} = \frac{-18-4i}{-18+2i} =$$

$$\underset{\downarrow}{=} \frac{-18-4i}{-18+2i} \cdot \frac{-18-2i}{-18-2i} = \frac{324+72i+36i-8}{324+4} = \frac{316}{328} + i\,\frac{108}{328} =$$

$\overline{-18+2i} = \overline{-18} + \overline{2i} = -18-2i$

$$= \frac{79}{82} + i\,\frac{27}{82}$$