Week 2

Lecture 4

In this lecture we shall discuss the basic properties of functions of one complex variable.

Functions of a Complex Variable

Def 2.1: Let $S \subseteq \mathbb{C}$ be a subset of complex numbers. A function f on S is a rule which assigns to each $z \in S$ a complex number (or complex numbers) $f(z)$, called the value of f at z. S is called the Domain of Definition (DoD)

Def 2.2: If for every z in the DoD corresponds one value w, then we say that f is single-valued function.
If to z correspond several values, then f is multi-valued function.

Ex 3.1 $f(z) = \frac{1}{z}$: f is not defined at $z=0$ and it is defined and single-valued everywhere else.

For example, the value of f at $z = 8-3i$ is

$$f(8-3i) = \frac{1}{8-3i} = \frac{1}{8-3i} \cdot \frac{8+3i}{8+3i} = \frac{8}{8^2+3^2} + i\,\frac{3}{8^2+3^2} = \frac{8}{73} + i\,\frac{3}{73}$$

$$f(i) = \frac{1}{i} = \frac{1}{i}\cdot\frac{-i}{-i} = \frac{-i}{1} = -i$$

Ex 3.2: $f(z) = \sqrt{z+1}$ defined everywhere on $\mathbb{C}$ and it is double-valued function: $f(0)=1$ and $f(0)=-1$ (for instance)

$f(0) = \sqrt{1}$: $|1| = 1$, $\cos\theta = \frac{1}{1}$, $\sin\theta = \frac{0}{1}$ $\Rightarrow \theta = 0 + 2\pi k$

$$\Rightarrow \sqrt{1} = \cos\left(\frac{2\pi k}{2}\right) + i\sin\left(\frac{2\pi k}{2}\right) \quad k = 0,1$$

$k=0$: $\cos 0 + i\sin 0 = 1$, $k=1$: $\cos\pi + i\sin\pi = -1$

Ex 3.3: $f(z) = \sqrt{z-1} + \sqrt{z+1}$ is 4-valued: for example, if $z=0$, we get $w_1 = i+1$, $w_2 = -i+1$, $w_3 = i-1$, $w_4 = -i-1$ (CHECK!)

Ex 3.4: Polynomials General form: $P_n(z) = \sum_{j=0}^{n} a_j z^j$ – polynomial of degree n of complex variable z with complex coefficients $a_j \in \mathbb{C}$.

For example: $f(z) = z^2$, $f(z) = 1 + iz - z^2 + (1+i)z^3$

Ex 3.5: Rational function {ratio of two polynomials}. General form $f(z) = \dfrac{a_0 + a_1 z + \ldots + a_n z^n}{b_0 + b_1 z + \ldots + b_m z^m}$ – defined and single-valued everywhere, except for the roots (zeros) of the denominator

Ex 3.6: The exponential function $f(z) = e^z$

Defined to be the series $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \dots$
or equally well by Euler's formula: $e^z = e^x(\cos y + i \sin y)$
$\{$ for $z = x+iy \}$
We will see later that the above series converges absolutely for all $z \in \mathbb{C}$, thus $f(z) = e^z$ can be defined for every z in $\mathbb{C}$, namely, we can take the DoD to be $\mathbb{C}$. It is single-valued.

Ex 3.7 Trigonometric and Hyperbolic functions: For $z \in \mathbb{C}$ define: (*) $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$ $\quad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$
which agrees with the definition for real z (by Euler's formula)
By analogy with the real case, we define

(**) $\cosh z = \frac{1}{2}(e^z + e^{-z})$, $\quad \sinh z = \frac{1}{2}(e^z - e^{-z})$

Using (*) and (**) it is easy to check that
CHECK! $\cos z + i \sin z = e^{iz} = \cosh(iz) + \sinh(iz)$

Let $z = x+iy$. We rewrite $f(z)$ and separate its real and imaginary parts as follows:
(1) $f(z) = \operatorname{Re} f(z) + i \operatorname{Im} f(z) = u(x,y) + i v(x,y)$
In this way we can think of f as of a real mapping:
$f: \mathbb{R}^2 \to \mathbb{R}^2$: (2) $(x,y) \mapsto (u(x,y), v(x,y))$ $\quad u, v: \mathbb{R}^2 \to \mathbb{R}^2$
Let $z = r(\cos\theta + i\sin\theta)$ $\{$ the polar form $\}$. Then, in a similar way
$f: \mathbb{R}^2 \to \mathbb{R}^2$: (3) $f(r,\theta) = \Psi(r,\theta) + i\varphi(r,\theta)$

Ex 3.8: Write in the form (1) or (3)
a) $f(z) = z^2$ $\quad$ b) $f(z) = |z|$ $\quad$ c) $f(z) = \operatorname{Re} z$ $\quad$ d) $f(z) = z^n$ $\quad$ e) $f(z) = e^z$

Sol: a) Denote $z = x+iy$. Then $z^2 = (x+iy)^2 = (x^2-y^2) + i2xy$
$\Rightarrow \operatorname{Re} z^2 = x^2 - y^2$, $\operatorname{Im} z^2 = 2xy$, and $f(x,y) = (x^2-y^2, 2xy)$
OR: $z = r(\cos\theta + i\sin\theta) \Rightarrow z^2 = r^2(\cos 2\theta + i \sin 2\theta)$, where $r = |z|$,
thus $\operatorname{Re} z^2 = r^2\cos 2\theta$, $\operatorname{Im} z^2 = r^2 \sin 2\theta$, and $f(r,\theta) = (r^2\cos 2\theta, r^2 \sin 2\theta)$

b) $|z| = |x+iy| = \sqrt{x^2+y^2} \Rightarrow f(x,y) = (\sqrt{x^2+y^2}, 0) = \sqrt{x^2+y^2} \rightarrow$ real function!

c) $z = x+iy$, $\operatorname{Re} z = x \Rightarrow f(x,y) = (x,0) = x \rightarrow$ real function!
$z = x+iy$, $\operatorname{Im} z = y \Rightarrow$ for $f(z) = \operatorname{Im} z$: $f(x,y) = (y,0) = y \rightarrow$ real!

d) Let $z = r(\cos\theta + i\sin\theta)$ $\{ r = |z| \}$. Then, by De Moivre formula

$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$, and $f(r,\theta) = (r^n\cos(n\theta), r^n\sin(n\theta))$

e) Let $z = x+iy$. Then, by Euler's formula: $e^z = e^x(\cos y + i\sin y)$
$\Rightarrow f(x,y) = (e^x\cos y, e^x\sin y)$

We can also do the opposite: given a function of x,y we can rewrite it in terms of z and its conjugate:

Recall: $\operatorname{Re} z = \frac{1}{2}(z+\bar{z})$ $\quad \operatorname{Im} z = \frac{1}{2i}(z-\bar{z})$

Ex 3.9: Rewrite in terms z and $\bar{z}$:

a) $f(x,y) = x^2 - y + i(x+y^2)$ $\qquad$ b) $f(x,y) = x^2 - 1$

Sol: a) $x^2 - y + i(x+y^2) = \frac{1}{4}(z+\bar{z})^2 - \frac{1}{2i}(z-\bar{z}) + i\left(\frac{1}{2}(z+\bar{z}) + \right.$
$\left. + \frac{1}{4i^2}(z-\bar{z})^2\right) = \frac{1}{4}(z^2+\bar{z}^2+2z\bar{z}) - \frac{1}{2i}(z-\bar{z}) + i\left(\frac{1}{2}(z+\bar{z})\right.$
$\left. - \frac{1}{4}(z^2+\bar{z}^2-2z\bar{z})\right) = \frac{1}{4}(z^2+\bar{z}^2+2z\bar{z}) + \frac{i}{2}(z-\bar{z}) + \frac{i}{2}(z+\bar{z})$

$\frac{1}{i} = -i$

$- \frac{i}{4}(z^2+\bar{z}^2-2z\bar{z}) = \frac{1}{4}(z^2+\bar{z}^2+2z\bar{z}) + iz - \frac{i}{4}(z^2+\bar{z}^2-2z\bar{z})$
$= \frac{1}{4}(z^2+\bar{z}^2)(1-i) + \frac{1}{2}z\bar{z}(1+i) + iz$

b) $x^2 - 1 = \frac{1}{4}(z^2+\bar{z}^2+2z\bar{z}) - 1 = \frac{1}{4}(z^2+\bar{z}^2+2z\bar{z}-4)$

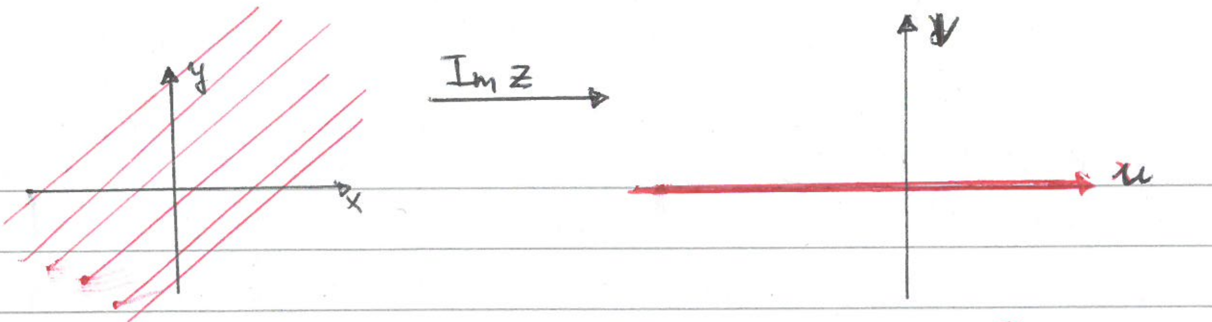
Transformations of the Complex plane

A complex function is a map (transformation) from $\mathbb{C}$ to $\mathbb{C}$, but to draw a graph of such mapping we would need $2+2=4$ real dimensions.

Instead, we shall examine what happens to various shapes - lines, circles, etc, under such map. Let us assume that f is single-valued function defined on some domain of definition. We will treat the function as a map that maps a set from (x,y)-plane to some other set in (u,v)-plane. Let us start with two simple examples.

Ex 4.1: Let $f(z) = \operatorname{Im} z$ (defined and single-valued everywhere). Let us find its image {namely, where to the complex plane $\mathbb{C}$ is mapped}

$\operatorname{Im} z = y \in \mathbb{R} \Rightarrow$ the complex plane $\mathbb{C}$ is mapped to $\mathbb{R}$:



Ex 4.2: What is the image of the ray $\arg z = \frac{\pi}{3}$ under $f(z)=z^3$?

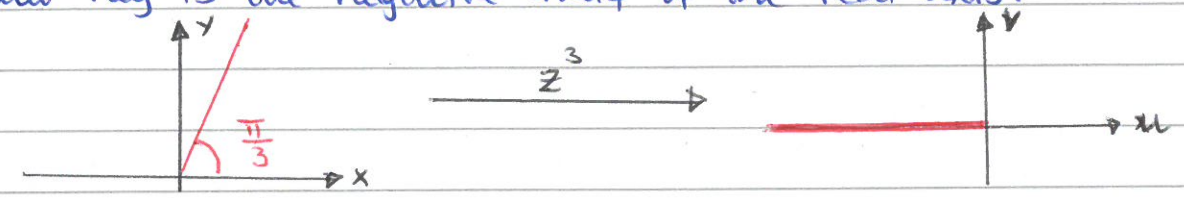
Sol: Let us write the function f in polar coordinates:

$$f(z)=r^3(\cos\theta+i\sin\theta)^3 \underset{\text{De Moivre}}{=} r^3(\cos 3\theta+i\sin 3\theta) \quad \{ r=|z| \}$$

Thus, the image of $\arg z = \frac{\pi}{3}$ under z^3 is

$$r^3\left(\cos\left(3\tfrac{\pi}{3}\right)+i\sin\left(3\tfrac{\pi}{3}\right)\right) = -r^3 \quad (r \geq 0!)$$

Recall that $r=|z|$, therefore $r\geq 0$ always. Hence, the image of that ray is the negative half of the real axis:



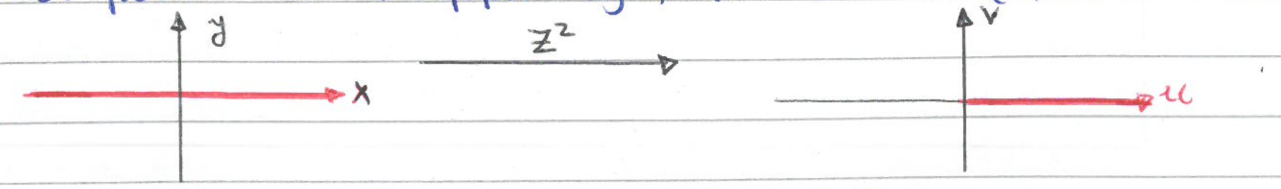
$f(z)=z^2$

Now we will examine the mapping $f(z)=z^2$ and will find the image of various curves and shapes under this mapping. Let us rewrite f in the polar and algebraic form:

$z=x+iy$: $f(x,y)=u(x,y)+iv(x,y)=x^2-y^2+i2xy$

$z=re^{i\theta}$: $f(r,\theta)=r^2e^{2i\theta}=r^2\cos 2\theta+ir^2\sin 2\theta$

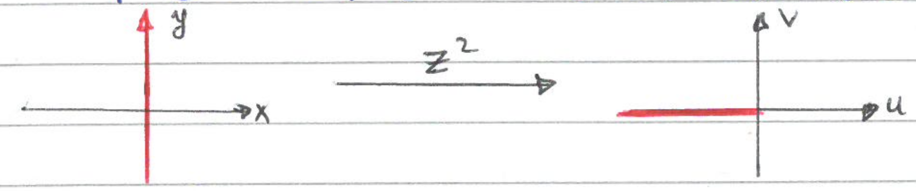
Ex 4.3: First, we determine what is the image of x-axis and of y-axis. The real axis $\mathbb{R}$ is mapped to the positive half of the real axis $\mathbb{R}_+$: $\mathbb{R}\xrightarrow{z^2}\mathbb{R}_+$ To see that we represent $\mathbb{R}$ as $\mathbb{R}=\mathbb{R}_+\cup\mathbb{R}_-$, where $\mathbb{R}_+$ ($\mathbb{R}_-$) the positive (negative) half of $\mathbb{R}$. Every point in $\mathbb{R}_+$ is of the form $re^{i0}=r>0$ and every point in $\mathbb{R}_-$ is of the form $re^{i\pi}=-r\leq 0$. These points are mapped by f to: $r^2e^{2\cdot i\cdot 0}=(-r)^2e^{2\cdot i\cdot\pi}=r^2\in\mathbb{R}_+$



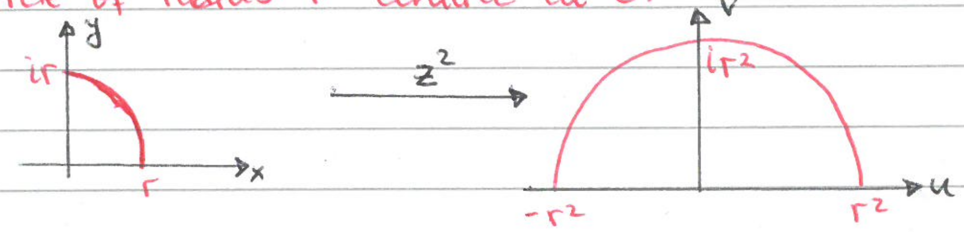
Now we will show (in the same way) that the imaginary axis is mapped to the negative half of the real axis: $i\mathbb{R}\xrightarrow{z^2}\mathbb{R}_-$

Represent the imaginary axis $i\mathbb{R}$ as: $i\mathbb{R} = (i\mathbb{R})_+ \cup (i\mathbb{R})_-$ where $(i\mathbb{R})_+$ are points of the form $re^{i\frac{\pi}{2}} = ir\ (r>0)$ and $(i\mathbb{R})_-$: $re^{i\frac{3\pi}{2}} = -ir$. f maps these points to

$$r^2 e^{i\frac{\pi}{2}\cdot 2} = r^2 e^{i\frac{3\pi}{2}\cdot 2} = -r^2 < 0 \quad (e^{i\pi} = e^{i3\pi} = -1)$$

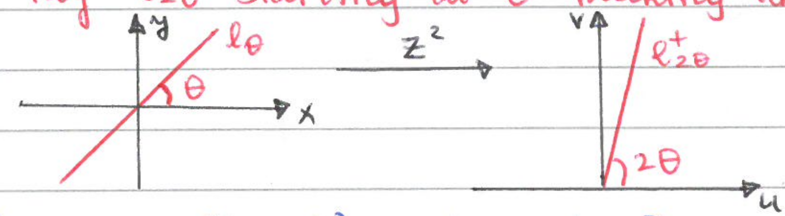


A quarter circle of radius r centered at 0 is mapped by z^2 to a half circle of radius r^2 centered at 0:



Pf: The quarter circle is given by points $re^{i\theta}$ where $0 \le \theta \le \frac{\pi}{2}$. It is mapped by z^2 to $r^2 e^{2i\theta}$. Since $0 \le \theta \le \frac{\pi}{2} \Rightarrow 0 \le 2\theta \le \pi$, thus we obtain a half circle of radius r^2 centered at 0. ☒

A line l_θ passing through the origin at angle θ is mapped to a ray $l^+_{2\theta}$ starting at 0 making an angle 2θ



The line is parametized by $re^{i\theta}$, where θ is fixed and r may vary.

The map $f(z) = z^2$ sends r to r^2 resulting in a ray and the angle is transformed from θ to 2θ.

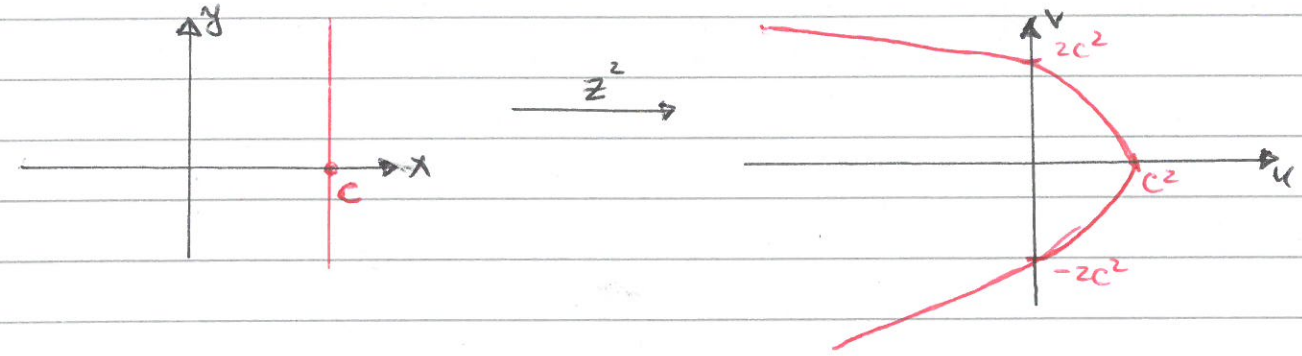
Now let us find the image of a vertical line $x = c \in \mathbb{R}$:

In this case the geometric representation is more convenient:

$z \to z^2 : (x,y) \mapsto (x^2 - y^2, 2xy)$. Set $x = c$

We have the following system, from which we may eliminate the variable y to get an equation for a curve in (u,v)-plane:

$$\begin{cases} u = x^2 - y^2 = c^2 - y^2 \\ v = 2xy = 2cy \end{cases} \Rightarrow y = \frac{v}{2c} \Rightarrow u = c^2 - \frac{v^2}{4c^2}$$: this is leftward opening parabola with vertex $(c^2, 0)$, intersecting the v-axis at $(0, \pm 2c^2)$ {Note that: if c is negative, the image again is the same, namely, the image is independent of the sign of c}



y
x
c
z²
v
2c²
c²
-2c²
u

Lectures 5+6

We continue to examine the images under z^2 and other transformations.

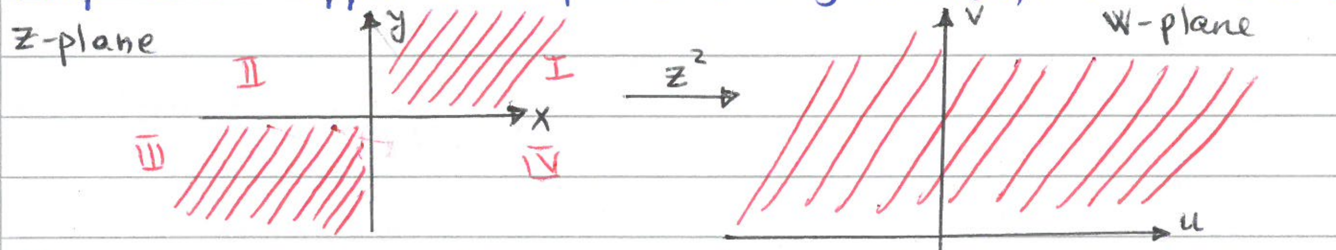
Ex 4.4: Determine which regions (if any) of the z-plane are mapped to the upper half part of the w-plane under the mapping $w = z^2$

First, we identify: the upper half of w-plane $= \{ w \in \mathbb{C} \mid \operatorname{Im} w \geq 0 \}$

An element $z = re^{i\theta} \xrightarrow{z^2} w = r^2 e^{2i\theta}$ is in the upper half plane iff the argument of w lies between 0 and π $\quad 0 \leq \arg w \leq \pi$:

$$0 + 2\pi k \leq 2\theta \leq \pi + 2\pi k, \; k \in \mathbb{Z} \;\Rightarrow\; k\pi \leq \theta \leq \frac{\pi}{2} + \pi k$$

Namely, z must lie in the first or third quadrant in order to map to the upper half plane [Plug $k = 1, 2, 3, 4$ to see this!]



Alternatively: we could use the algebraic form $x+iy$ and get:

$w = z^2 = x^2 - y^2 + i\,2xy \Rightarrow \operatorname{Im} w \geq 0 \Leftrightarrow 2xy \geq 0 \Leftrightarrow x, y \geq 0$ or $x, y \leq 0$

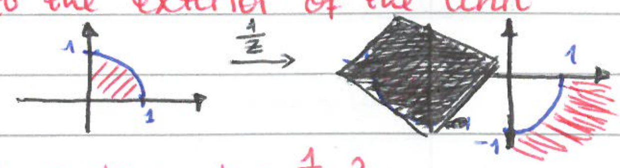
This gives again I and III quadrant

Now let us study another function and its images.

Ex 4.5 Look at $z \to \frac{1}{z} = w$

It is an instructive exercise to draw a picture trying to determine where to various regions of the plane are mapped.

EXERCISE: 1) Show that $w = \frac{1}{z}$ maps the interior of the unit circle in the first quadrant to the exterior of the unit circle in the fourth quadrant:



2) What is the image of the unit circle under $\frac{1}{z}$?

3) Write down how $z \to \frac{1}{z} = w$ maps the punctured plane $\mathbb{C} \setminus \{0\}$ to itself (and $\mathbb{C} \cup \{\infty\}$ to itself)

Back to Ex 4.5. Let us study two images under $\frac{1}{z}$

a) Find the image of the line $z = t + i(1-t)$ under $\frac{1}{z}$.

Sol: Denote: $w = u + iv$, $z = x + iy \Rightarrow x = t$, $y = (1-t)$

$w = u + iv = \frac{1}{x+iy} = \frac{1}{z}$ (for $z \neq 0$, $w \neq 0$) $\Leftrightarrow$ $x+iy = \frac{1}{u+iv} = \frac{u}{u^2+v^2} + i\frac{-v}{u^2+v^2}$

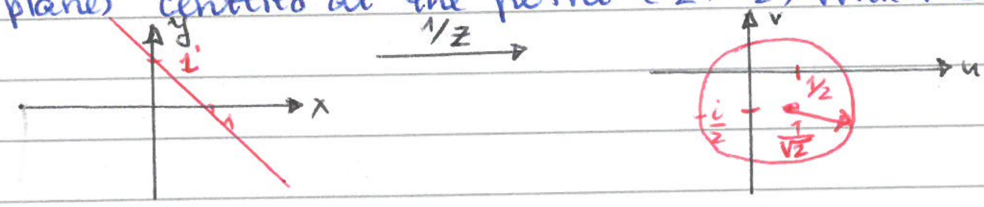
Thus, plugging $x = t$, $y = 1-t$, we obtain the following system:

$$\begin{cases} t = \frac{u}{u^2+v^2} \\ 1-t = \frac{-v}{u^2+v^2} \end{cases} \Leftrightarrow \begin{cases} t(u^2+v^2) = u \\ (1-t)(u^2+v^2) = -v \end{cases} \Leftrightarrow \begin{cases} t(u^2+v^2) = u \\ u^2+v^2 - t(u^2+v^2) = -v \end{cases}$$

$\Leftrightarrow$ $u^2+v^2 = u - v$ (we add the equations) $\Leftrightarrow$ $u^2 - u + v^2 + v = 0$

$\Leftrightarrow$ $(u-\frac{1}{2})^2 + (v+\frac{1}{2})^2 = \frac{1}{2}$ (complete the square in u and in v)

Namely, the map $w = \frac{1}{z}$ maps the line $z = t + i(1-t)$ to the circle (in w-plane) centered at the point $(\frac{1}{2}, -\frac{1}{2})$ with radius $\frac{1}{\sqrt{2}}$:



b) Determine how the horizontal lane $y = k$ is transformed under $w = \frac{1}{z}$

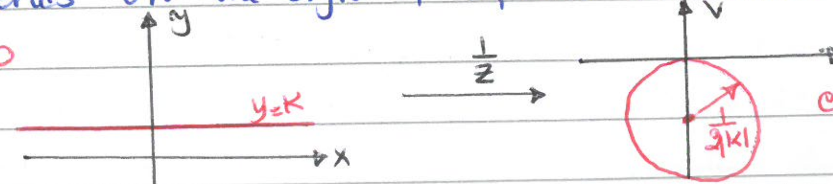
Sol: Substitute as before: $x = \frac{u}{u^2+v^2}$, $y = \frac{-v}{u^2+v^2}$. Plug $y = k$:

$k = \frac{-v}{u^2+v^2}$ $\Leftrightarrow$ $ku^2 + kv^2 + v = 0$ $\Leftrightarrow$ $(k \neq 0)$ $u^2 + v^2 + \frac{v}{k} = 0$

$\Leftrightarrow$ $u^2 + (v + \frac{1}{2k})^2 = \frac{1}{4k^2}$ (complete the square in v)

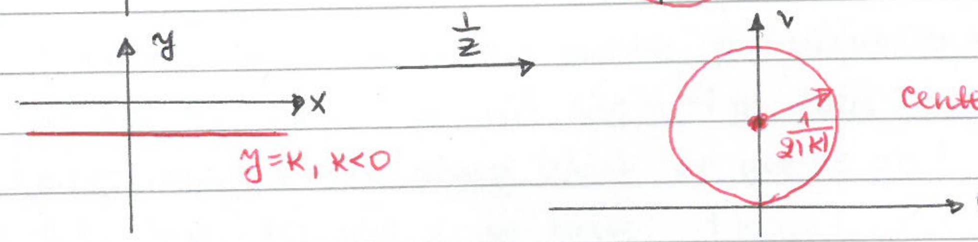
Namely, the line is mapped onto a circle, centre of which depends on the sign of k, and the radius is $\frac{1}{2|k|}$

$k > 0$



center $= (0, -\frac{1}{2k})$

$k < 0$



center $= (0, \frac{1}{2k})$

Now we have a feeling on geometrical description of the complex functions; we will study more transformations later.
Next subject - the limits of complex functions. The introduction of a limit will allow us to investigate what it means for the complex function f to be continuous and differentiable.
Through several explicit examples, we contrast our findings

with corresponding results for real functions.

Limits

Let f be a complex function. We say that $\lim_{z\to z_0} f(z) = w_0$ if when z is "close" to z_0, $f(z)$ is "close" to w_0. Formally:

Def 3.1: $\lim_{z\to z_0} f(z) = w_0$ if for any given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $0 < |z - z_0| < \delta$, then $|f(z) - w_0| < \varepsilon$

Ex 5.1: Let $f(z) = \frac{\bar{z}}{2}$. Claim: for any z_0 $\lim_{z\to z_0} \frac{\bar{z}}{2} = \frac{\bar{z}_0}{2}$

Pf: Given $\varepsilon > 0$, choose $\delta = 2\varepsilon$. Then, if $|z - z_0| < \delta$:

$$\left|\frac{\bar{z}}{2} - \frac{\bar{z}_0}{2}\right| = \frac{1}{2}|\bar{z} - \bar{z}_0| = \frac{1}{2}|z - z_0| < \frac{1}{2}\delta = \frac{1}{2}2\varepsilon = \varepsilon$$

Thus, for a given $\varepsilon > 0$ we have found δ (that depends on ε!) s.t. if $|z - z_0| < \delta$, then $|f(z) - w_0| = \left|\frac{\bar{z}}{2} - \frac{\bar{z}_0}{2}\right| < \varepsilon$ (for any z_0!)

Ex 5.2: Let $f(z) = \frac{\bar{z}}{z}$, $z \neq 0$. Prove that $\lim_{z\to 0} \frac{\bar{z}}{z}$ does not exist.

Before we proceed to the proof, let us show the following:

Claim 3.1: If $\lim_{z\to z_0} f(z)$ exists, then it is unique.

Pf: Assume $\lim_{z\to z_0} f(z) = a$ and $\lim_{z\to z_0} f(z) = b$. By Def 3.1 for any $\varepsilon > 0$ there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

if $|z - z_0| < \delta_1 \Rightarrow |f(z) - a| < \varepsilon/2$

if $|z - z_0| < \delta_2 \Rightarrow |f(z) - b| < \varepsilon/2$

Choose $\delta < \min(\delta_1, \delta_2) \Rightarrow$ for $0 < |z - z_0| < \delta$ $|f(z) - a| < \varepsilon/2$ and $|f(z) - b| < \varepsilon/2$. Thus, using triangle inequality, we obtain

(*) $|a - b| = |a - f(z) + f(z) - b| \leq |a - f(z)| + |f(z) - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Since (*) holds for any $\varepsilon > 0$ we obtain that $a = b$. ∎

Back to Ex 5.2: we will use a common technique to show that the limit does not exist – we will show that there exist two different paths (directions) $z \to z_0$ along which we get 2 different limits

Ex 5.2 Pf: Note: for real z we have: {namely, fix $y = 0$}

$z = x \in \mathbb{R}$: $\frac{\bar{z}}{z} = \frac{\bar{x}}{x} = \frac{x}{x} = 1 \Rightarrow \lim_{x\to 0} \frac{\bar{z}}{z} = 1$.

On the other hand, for purely imaginary z we get: {namely, fix $x = 0$}

$z = iy \in i\mathbb{R}$: $\frac{\bar{z}}{z} = \frac{\overline{iy}}{iy} = \frac{-iy}{iy} = -1 \Rightarrow \lim_{iy\to 0} \frac{\bar{z}}{z} = -1$

$\Rightarrow$ the limit as $z \to 0$ does not exist. ∎

Note: For any $z_0 \neq 0$ $\lim_{z\to z_0} \frac{\bar{z}}{z} = \frac{\bar{z}_0}{z_0}$

Pf: Given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{2}|z_0|$. Then

$$\left|\frac{\bar z}{z}-\frac{\bar z_0}{z_0}\right| = \left|\frac{\bar z z_0 - \bar z_0 z}{z z_0}\right| = \left|\frac{\bar z z_0 + \bar z z - \bar z z - \bar z_0 z}{z z_0}\right| = \left|\frac{\bar z(z_0 - z) + z(\bar z - \bar z_0)}{z z_0}\right|$$

$$\leq \left|\frac{\bar z(z_0-z)}{z z_0}\right| + \left|\frac{z(\bar z - \bar z_0)}{z z_0}\right| = \frac{|\bar z||z_0 - z|}{|z||z_0|} + \frac{|z||\bar z - \bar z_0|}{|z||z_0|} =$$

$$= \frac{|z||z_0-z|}{|z||z_0|} + \frac{|z-z_0|}{|z_0|} = 2\frac{|z-z_0|}{|z_0|} < \frac{2}{|z_0|}\delta = \frac{2}{|z_0|}\cdot\frac{\varepsilon}{2}|z_0| = \varepsilon \quad \blacksquare$$

Ex 5.3 Show that $\lim_{z\to 0}\frac{z^2}{\bar z^2}$ does not exist

Sol: Here, for $z = x\in\mathbb{R}$: $\frac{z^2}{\bar z^2} = \frac{x^2}{\bar x^2} = \frac{x^2}{x^2} = 1$ and for $z = iy \in i\mathbb{R}$ $\frac{z^2}{\bar z^2} = \frac{(iy)^2}{(\bar{iy})^2} = \frac{-y^2}{-y^2} = 1$. However, choosing the path $z = (1+i)t$ for $t\in\mathbb{R}\setminus\{0\}$ gives us $\frac{z^2}{\bar z^2} = \frac{2it^2}{-2it^2} = -1 \neq 1$ – does not agree with the value on previous two paths, thus we can conclude that the original limit does not exist

Alternatively: Let $z = re^{i\theta}$, $r,\theta\in\mathbb{R}$. Then $z\to 0$ means $r\to 0$ since $e^{i\theta}\neq 0$ for all $\theta\in\mathbb{R}$. We have: $\frac{z^2}{\bar z^2} = \frac{r^2e^{2i\theta}}{r^2e^{-2i\theta}} = e^{4i\theta}$

$\Rightarrow \lim_{z\to 0}\frac{z^2}{\bar z^2} = \lim_{r\to 0}\frac{r^2e^{2i\theta}}{r^2e^{-2i\theta}} = \lim_{r\to 0} e^{4i\theta} = e^{4i\theta}$

For different θ-s we get different limits – namely, on different paths the limit does not agree $\Rightarrow$ the limit does not exist.

Ex 5.3 shows that the different paths need not be the real and imaginary axes!

Rmk: If $\lim_{z\to z_0} f(z) = w_0$ exists, then $f(z)\to w_0$ as $z\to z_0$ on every path (uniqueness of the limit!) If on different paths $z\to z_0$ there are different values of the limit (or there exists a path on which the limit does not exist), then the limit $\lim_{z\to z_0} f(z)$ does not exist.

Similarly to the real functions we have the following proposition for the complex functions.

Prop 3.1: If $\lim_{z\to z_0} f(z) = w_0$ and $\lim_{z\to z_0} g(z) = w_1$, then

1) $\lim_{z\to z_0}(f(z)+g(z)) = w_0 + w_1$
2) $\lim_{z\to z_0} f(z)\cdot g(z) = w_0\cdot w_1$
3) If $w_1\neq 0$, then $\lim_{z\to z_0}\frac{f(z)}{g(z)} = \frac{w_0}{w_1}$

Pf: EXERCISE – a very good exercise to "play" with ε-δ definition!

Next proposition provides a relation between the limit of the complex function and its real and imaginary parts (that are real functions)

Prop 3.2: Let $f(z) = u(x,y) + iv(x,y)$, $z_0 = x_0 + iy_0$. Then
$\lim_{z \to z_0} f(z) = u_0 + iv_0 = w_0 \iff \lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 = \operatorname{Re} w_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0 = \operatorname{Im} w_0$

Note: the first limit is complex, but the last two limits are real!

Pf: $\boxed{\Rightarrow}$ Assume that $\lim_{z\to z_0} f(z) = u_0 + iv_0$. Let us show that
$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 = \operatorname{Re} w_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0 = \operatorname{Im} w_0$.

By Def 3.1: given $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ s.t. for all z with $0 < |z - z_0| < \delta$ we have $|f(z) - (u_0 + iv_0)| < \varepsilon$.

Since for any $z \in \mathbb{C}$: $|\operatorname{Im} z|, |\operatorname{Re} z| < |\operatorname{Re} z + i \operatorname{Im} z| = |z|$, we get:

$$|v(x,y) - v_0|, \ |u(x,y) - u_0| < |u(x,y) - u_0 + i(v(x,y) - v_0)|$$
$$= |u(x,y) + iv(x,y) - (u_0 + iv_0)|$$
$$= |f(z) - (u_0 + iv_0)| < \varepsilon$$

Thus $|u(x,y) - u_0| < \varepsilon$ and $|v(x,y) - v_0| < \varepsilon$

CHECK: $|z - z_0|$ is the distance from (x,y) to (x_0, y_0) in $\mathbb{R}^2$

$\Rightarrow \lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$

Note: $|z - z_0| < \delta \Rightarrow |x + iy - (x_0 + iy_0)| = |(x - x_0) + i(y - y_0)| < \delta$, and we get $|x - x_0|, |y - y_0| \leq |(x - x_0) + i(y - y_0)| < \delta$

$\boxed{\Leftarrow}$ Assume that $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$ (*)

Let $\varepsilon > 0$. Since u and v have real limits u_0, v_0 as $(x,y) \to (x_0, y_0)$ from Def 3.1 there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that if
$0 < \|(x,y) - (x_0,y_0)\| < \delta_1$, then $|u(x,y) - u_0| < \varepsilon/2$, and
$0 < \|(x,y) - (x_0,y_0)\| < \delta_2$, then $|v(x,y) - v_0| < \varepsilon/2$.

Choose $\delta < \min(\delta_1, \delta_2)$. Then, whenever $0 < \|(x,y) - (x_0,y_0)\| < \delta$:

$$|u(x,y) + iv(x,y) - (u_0 + iv_0)| = |u(x,y) - u_0 + i(v(x,y) - v_0)|$$
$$\underset{\text{triangle inequality}}{\leq} |u(x,y) - u_0| + |v(x,y) - v_0| \underset{\text{by (*)}}{<} \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Namely, for any $\varepsilon > 0$ we have found $\delta > 0$ such that whenever $0 < |z - z_0| < \delta$: $|f(z) - (u_0 + iv_0)| < \varepsilon$, as required. ▨