Problem Set 1. Solutions

1) a) We know that $z_1 = z_2 \Leftrightarrow \text{Re}\, z_1 = \text{Re}\, z_2$ and $\text{Im}\, z_1 = \text{Im}\, z_2$.

We have $4a + (3-2b)i = 13+3i$, namely

$$4a - 2bi = 13 \Leftrightarrow 4a = 13 \text{ and } 2b = 0 \Leftrightarrow a = \frac{13}{4},\ b = 0$$

The second in the same way:

$$4(5-2a) - (b-4)i = 10-5i \Leftrightarrow -8a - bi = -10-9i$$

$\Leftrightarrow -8a = -10$ and $-b = -9$, thus $a = \frac{10}{8}$, $b = 9$

b) $(-10-2i)(4+5i) = -40-8i-50i-10i^2 \underset{i^2=-1}{=} -40-58i+10$

$= -30-58i$

$$(-3+7i)^2 - (2+i)^2 = 9+49i^2-42i-(4+i^2+4i)$$
$$= 9-49-42i-4+1-4i = -43-46i$$

$(i+i^2+i^3+\ldots+i^{157})^{37}$:

First, let us note the following

$i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, $i^6 = -1$, $i^7 = -i$, $i^8 = 1$ and so on.

Thus, we see that for any $k \in \mathbb{Z}$:

$$i^k + i^{k+1} + i^{k+2} + i^{k+3} = i^k(1+i+i^2+i^3) = i^k(1+i-1-i) = 0$$

We have 157 elements in the sum. The sum of the first 156 is 0, thus we obtain

$$(i+i^2+i^3+\ldots+i^{157})^{37} = (i^{157})^{37} \underset{\text{since } i^{157} = i \text{ (why?)}}{=} i^{37}$$

Let us compute the value of i^{37} using De Moivre formula:

$|i| = 1$ $\cos\theta = \frac{0}{1} = 0$ $\sin\theta = \frac{1}{1} = 1$ $\Rightarrow \theta = \frac{\pi}{2}$ $(+2\pi k,\ k \in \mathbb{Z})$

$\Rightarrow i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \Rightarrow i^{37} = \cos\left(\frac{\pi}{2}37\right) + i\sin\left(\frac{\pi}{2}37\right) =$

$= \cos\left(\frac{\pi}{2}+18\pi\right) + i\sin\left(\frac{\pi}{2}+18\pi\right) = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$

Alternatively: $i^{37} = i(i^{36}) = i\cdot(i^4)^9 = i\cdot(1)^9 = i$

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c) 1) $(2z-i)(3+2i) = (z+i)(1-3i)$

$\Leftrightarrow 6z - 3i + 4iz + 2 = z + i - 3iz + 3$

$\Leftrightarrow 5z + 7iz = 1+4i$

$\Leftrightarrow z(5+7i) = 1+4i \Leftrightarrow z = \frac{1+4i}{5+7i}\cdot\frac{5-7i}{5-7i} = \frac{33}{74} + i\cdot\frac{13}{74}$

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2) $\frac{1-iz}{1+iz}=-2i \iff 1-iz=-2i(1+iz) \iff 1-iz=-2i+2z$

$\iff 2z+iz=1+2i \iff z(2+i)=1+2i$

$\iff z=\frac{1+2i}{2+i}=\frac{1+2i}{2+i}\cdot\frac{2-i}{2-i}=\frac{2+4i-i+2}{2^2+1^2}=\frac{4+3i}{5}=\frac{4}{5}+i\frac{3}{5}$

3) $i(3z-1)=3z\bar{z}$

Denote $z=x+iy$, then $\bar{z}=x-iy$ and $z\bar{z}=x^2+y^2$. Thus, we have

$i(3x+3iy-1)=3(x^2+y^2)$

$\iff 3ix-3y-i-3x^2-3y^2=0 \iff 3x^2+3y^2+3y+i(1-3x)=0$

Since for any $z\in\mathbb{C}$ $z=0 \iff \operatorname{Re} z=0$ and $\operatorname{Im} z=0$ we obtain the following system of equations

$$\begin{cases} 3x^2+3y^2+3y=0 \\ 1-3x=0 \end{cases} \Rightarrow 3x=1 \Rightarrow x=\frac{1}{3}$$

Plug $x=\frac{1}{3}$ into the first equation:

$$3\cdot\frac{1}{9}+3y^2+3y=0 \iff y^2+y+\frac{1}{9}=0 \Rightarrow y=-\frac{1}{2}\pm\frac{\sqrt{5}}{6}$$

Thus, we have two solutions

$x=\frac{1}{3}$ $\qquad$ $x=\frac{1}{3}$

$y=-\frac{1}{2}+\frac{\sqrt{5}}{6}$ $\qquad$ $y=-\frac{1}{2}-\frac{\sqrt{5}}{6}$

Namely,

$$z_1=\frac{1}{3}+i\left(-\frac{1}{2}+\frac{\sqrt{5}}{6}\right),\quad z_2=\frac{1}{3}+i\left(-\frac{1}{2}-\frac{\sqrt{5}}{6}\right)$$

d) 1) $\frac{1}{z}+\frac{1}{\bar{z}}=\frac{1}{z}\cdot\frac{\bar{z}}{\bar{z}}+\frac{1}{\bar{z}}\cdot\frac{z}{z}=\frac{\bar{z}+z}{z\bar{z}}$

$\overline{(\bar{z})}=z$

Let $z=x+iy$. Then $\bar{z}+z=x-iy+x+iy=2x=2\operatorname{Re} z\in\mathbb{R}$,

$z\bar{z}=(x+iy)(x-iy)=x^2+iyx-ixy-i^2y^2=x^2+y^2\in\mathbb{R}$

$\Rightarrow \frac{1}{z}+\frac{1}{\bar{z}}=\frac{\bar{z}+z}{z\bar{z}}=\frac{2x}{x^2+y^2}\in\mathbb{R}$

2) $z^3\bar{z}+z\bar{z}^3=z^2z\bar{z}+z\bar{z}\bar{z}^2=z^2|z|^2+\bar{z}^2|z|^2=|z|^2(z^2+\bar{z}^2)$

Since $|z|^2=x^2+y^2\in\mathbb{R}$ we are left to show that $z^2+\bar{z}^2\in\mathbb{R}$.

$z^2+\bar{z}^2=(x+iy)^2+(x-iy)^2=x^2-y^2+2ixy+x^2-y^2-2ixy=2(x^2-y^2)\in\mathbb{R}$

e) By Prop 1.2 and Prop 1.2½: for any $z, w \in \mathbb{C}$

$$|zw| = |z||w| \qquad \left|\frac{z}{w}\right| = \frac{|z|}{|w|} \quad (w \neq 0)$$

Thus, we have

$$\left|\frac{(3+5i)(1-2i)^6}{i^{20}(3+5i)^2}\right| = \left|\frac{(1-2i)^6}{i^{20}(3+5i)}\right| = \frac{|(1-2i)^6|}{|i^{20}||3+5i|} = \frac{|1-2i|^6}{|i|^{20}|3+5i|}$$

$$|1-2i| = \sqrt{1^2+(-2)^2} = \sqrt{5} \Rightarrow |1-2i|^6 = (\sqrt{5})^6 = 5^3 = 125$$

$$|i| = \sqrt{0^2+1^2} = 1 \Rightarrow |i|^{20} = 1$$

$$|3+5i| = \sqrt{3^2+5^2} = \sqrt{34}$$

$$\Rightarrow \left|\frac{(3+5i)(1-2i)^6}{i^{20}(3+5i)^2}\right| = \frac{125}{\sqrt{34}}$$

2) a) $-1+i$: We need to compute the modulus and the argument:

$$|-1+i| = \sqrt{(-1)^2+1^2} = \sqrt{2} \qquad \cos\theta = \frac{-1}{\sqrt{2}},\ \sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{3\pi}{4}\ (+2\pi k)$$

Thus, we obtain

$$-1+i = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

b) $|\sqrt{3}-i| = \sqrt{(\sqrt{3})^2+(-1)^2} = \sqrt{4} = 2 \quad \cos\theta = \frac{\sqrt{3}}{2},\ \sin\theta = \frac{-1}{2}$

$\Rightarrow \theta = \frac{11\pi}{6}\ (+2\pi k)$ and

$$\sqrt{3}-i = 2\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$$

c) $|4| = 4 \quad \cos\theta = \frac{4}{4} = 1,\ \sin\theta = \frac{0}{4} = 0 \Rightarrow \theta = 0\ (+2\pi k)$, and

$$4 = 4(\cos 0 + i\sin 0)$$

d) $|-i| = 1 \quad \cos\theta = \frac{0}{1} = 0 \quad \sin\theta = \frac{-1}{1} = (-1) \Rightarrow \theta = \frac{3\pi}{2}\ (+2\pi k)$, and

$$-i = 1\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$$

e) By Prop 1.2 and Prop 1.2½:

$$\left|\frac{-2}{1+i\sqrt{3}}\right| = \frac{|-2|}{|1+i\sqrt{3}|} = \frac{2}{\sqrt{1^2+(\sqrt{3})^2}} = \frac{2}{2} = 1.$$

To find the argument we need to compute the real and imaginary part of $\frac{-2}{1+i\sqrt{3}}$:

$$\frac{-2}{1+i\sqrt{3}} = \frac{-2}{1+i\sqrt{3}} \cdot \frac{1-i\sqrt{3}}{1-i\sqrt{3}} = \frac{-2+2i\sqrt{3}}{4} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$\Rightarrow \cos\theta = \frac{-1/2}{1} = -\frac{1}{2} \quad \sin\theta = \frac{\sqrt{3}/2}{1} = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{2\pi}{3}\ (+2\pi k)$, and

$$\frac{-2}{1+i\sqrt{3}} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$$

f) $|5+3i| = \sqrt{5^2+3^2} = \sqrt{34}$ $\quad \cos\theta = \frac{5}{\sqrt{34}}$ $\quad \sin\theta = \frac{3}{\sqrt{34}}$

$\Rightarrow \theta = \arccos\frac{5}{\sqrt{34}}$ (or $\arcsin\frac{3}{\sqrt{34}}$ – choose one!), and

$$5+3i = \sqrt{34}\left(\cos\left(\arccos\frac{5}{\sqrt{34}}\right) + i\sin\left(\arccos\frac{5}{\sqrt{34}}\right)\right)$$

3) a) $z = \sqrt[7]{1-i}$. First, write z in polar form:

$|1-i| = \sqrt{1^2+(-1)^2} = \sqrt{2}$; $\cos\theta = \frac{1}{\sqrt{2}}$, $\sin\theta = \frac{-1}{\sqrt{2}} \Rightarrow \theta = \frac{7\pi}{4} (+2\pi k)$

$\Rightarrow 1-i = \sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$

By the formula for n-th root:

$$\sqrt[7]{1-i} = (\sqrt{2})^{1/7}\left(\cos\left(\frac{\frac{7\pi}{4}+2\pi k}{7}\right) + i\sin\left(\frac{\frac{7\pi}{4}+2\pi k}{7}\right)\right) =$$

$$= 2^{\frac{1}{14}}\left(\cos\left(\frac{7\pi+8\pi k}{28}\right) + i\sin\left(\frac{7\pi+8\pi k}{28}\right)\right) \quad k=0,1,2,3,4,5,6$$

b) $z^{14}=i$, namely $z = \sqrt[14]{i}$. Write i in polar form:

$|i| = 1$ $\quad \cos\theta = \frac{0}{1} = 0$ $\quad \sin\theta = \frac{1}{1} = 1 \Rightarrow \theta = \frac{\pi}{2} + 2\pi k$

Thus,

$$\sqrt[14]{i} = 1^{1/14}\left(\cos\left(\frac{\frac{\pi}{2}+2\pi k}{14}\right) + i\sin\left(\frac{\frac{\pi}{2}+2\pi k}{14}\right)\right)$$

$$= \cos\left(\frac{\pi+4\pi k}{28}\right) + i\sin\left(\frac{\pi+4\pi k}{28}\right) \quad k=0,1,2,..,13$$

c) $z = \sqrt[6]{64}$: $64 = 64 + i\cdot 0$ – $|64| = 64$, $\cos\theta = \frac{64}{64} = 1$,

$\sin\theta = \frac{0}{64} = 0 \Rightarrow \theta = 0 + 2\pi k$

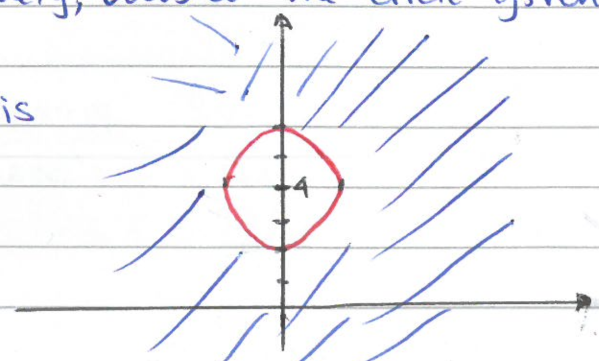
$$\sqrt[6]{64} = \sqrt[6]{64}\left(\cos\left(\frac{0+2\pi k}{6}\right) + i\sin\left(\frac{0+2\pi k}{6}\right)\right) = 2\left(\cos\frac{\pi k}{3} + i\sin\frac{\pi k}{3}\right)$$

for $k = 0,1,2,3,4,5$

4) a) The equation $|z-z_0| = R$ describes circle centered at z_0 of radius R, $|z-z_0| < R$ – the interior of that circle, and $|z-z_0| > R$ the exterior of that circle. Therefore, $|z-4i| > 2$ describes all the points that lie outside the circle centered at $(0,4i)$ of radius 2. Namely, outside the circle given by

$$x^2 + (y-4)^2 = 2$$

Note that the circle itself is also excluded!



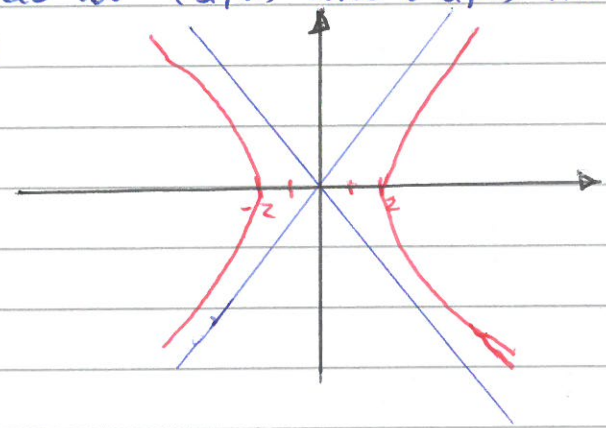
b) $\mathrm{Re}\, z^2 = 4$. Write z in algebraic form (Cartesian coordinates)

$z = x+iy$. Then $z^2 = (x+iy)^2 = x^2 - y^2 + i2xy$

$\Rightarrow \mathrm{Re}\, z^2 = x^2 - y^2$ and we have an equation $x^2 - y^2 = 4$

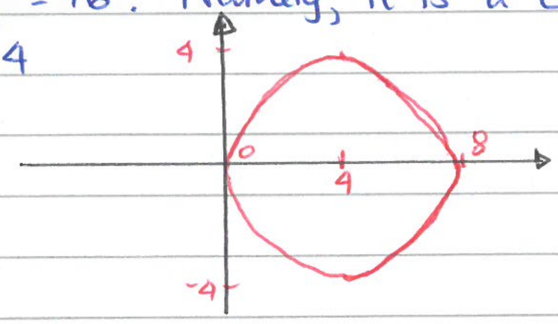
$\Leftrightarrow \frac{x^2}{4} - \frac{y^2}{4} = 1 \Leftrightarrow \left(\frac{x}{2}\right)^2 - \left(\frac{y}{2}\right)^2 = 1$

This is hyperbola with vertices at $(2,0)$ and $(-2,0)$ with asymptotes at $y = \pm \frac{2}{2} x = \pm x$



c) $\mathrm{Re}\, \frac{1}{z} = \frac{1}{8}$: Since $\mathrm{Re}\, \frac{1}{z} = \frac{x}{x^2+y^2}$ we get

$\frac{x}{x^2+y^2} = \frac{1}{8} \Leftrightarrow x^2+y^2-8x=0$. Complete the square in x and obtain: $(x-4)^2+y^2 = 16$. Namely, it is a circle centered at $(4,0)$ of radius 4

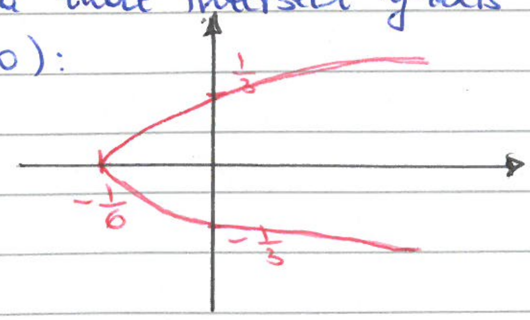


d) $|z| = \mathrm{Re}\, z + \frac{1}{3}$. Write $z = x+iy$, then

$\sqrt{x^2+y^2} = x + \frac{1}{3} \Leftrightarrow x^2+y^2 = x^2 + \frac{2}{3}x + \frac{1}{9}$

$\Leftrightarrow x = \frac{3}{2}y^2 - \frac{1}{6}$

This is rightward openning parabola that intersect y-axis at $(0, \pm\frac{1}{3})$ with vertex at $(-\frac{1}{6}, 0)$:



e) $|z+1| + |z-1| = 4$: Write $z = x+iy$. Then

$|x+iy+1| + |x+iy-1| = 4 \Leftrightarrow \sqrt{(x+1)^2+y^2} + \sqrt{(x-1)^2+y^2} = 4$

$\Rightarrow (x+1)^2+y^2 = \left(4 - \sqrt{(x-1)^2+y^2}\right)^2 = 16 + (x-1)^2+y^2 - 8\sqrt{(x-1)^2+y^2}$

$\Rightarrow (x+1)^2 - (x-1)^2 - 16 = -8\sqrt{(x-1)^2+y^2}$

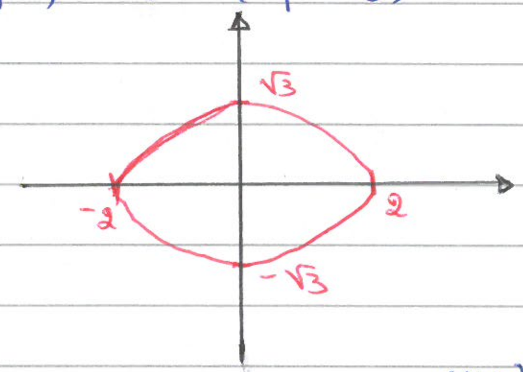
$\Rightarrow \quad 4x-16 = -8\sqrt{(x-1)^2+y^2} \quad \Rightarrow \quad (4x-16)^2 = 64\left((x-1)^2+y^2\right)$

$\Rightarrow \quad 16x^2-128x+256 = 64x^2-128x+64+64y^2$

$\Rightarrow \quad 48x^2+64y^2=192 \iff 3x^2+4y^2=12 \iff \frac{x^2}{4}+\frac{y^2}{3}=1$

$\iff \quad \frac{x^2}{2^2}+\frac{y^2}{(\sqrt{3})^2}=1$ - this is an ellipse with the focal points

at $(\pm 2, 0)$ and $(0, \pm\sqrt{3})$



Recall: Ellipse equation $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$, focal points are $(\pm a, 0)$, $(0, \pm b)$