Complex Variables.

Week 1.

Lecture 1.

What is the use of complex functions and complex variables? One of many things, for example, to compute the actual value of real convergent series. We know, for example, that the series $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$ convergent. The question is $\sum_{n=1}^{\infty} \frac{1}{n^2} = ?$ Towards the end of this module we will be able to compute the answer. We well see

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$$
$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \dots = \frac{\pi^4}{90}$$
$$1 + \frac{1}{2^6} + \frac{1}{3^6} + \dots = \frac{\pi^6}{245}$$

Using complex functions we will be able to compute quite complicated (real!) integrals, which may be very hard to compute otherwise and a lot of other things.

Compute: $\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = ?$

We start with the basic definitions, properties, and the geometry of complex numbers.

I The Field of Complex numbers.

Def 1.1 A complex number is an ordered pair of real numbers: (x,y). If $y=0$ we get the pair $(x,0)=x$ {we denote it by x}

Since real numbers are a part of complex numbers, the algebraic actions (sum, subtraction, multiplication, division) on complex numbers need to be such that they agree with the correspon-ding algebraic actions on real numbers We denote

$\mathbb{C}$ - the set of all complex numbers

We define the operations of addition and multiplication as follows.

Def 1.2: Two complex numbers $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, $z_1, z_2 \in \mathbb{C}$ are equal if and only if $x_1 = x_2$ and $y_1 = y_2$

Def 1.3: Sum of two complex numbers $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ is a complex number $z_1 + z_2 = (x_1 + x_2, y_1 + y_2) = z_3 \in \mathbb{C}$

Def 1.4: Product of two complex numbers $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$

is a complex number $z_1 \cdot z_2 = (x_1 \cdot x_2 - y_1 \cdot y_2,\ x_1 \cdot y_2 + x_2 \cdot y_1) = z_3 \in \mathbb{C}$

It is easy to see that if we apply this rule to real numbers, we get the usual product

For $x_1, x_2 \in \mathbb{R}$: $x_1 = (x_1, 0)$, $x_2 = (x_2, 0)$

$$(x_1,0)(x_2,0) = (x_1x_2 - 0\cdot 0,\ x_1 \cdot 0 + 0 \cdot x_1) = (x_1x_2, 0) = x_1x_2$$

Usually we will write:

For $z \in \mathbb{C}$: $z = x + iy$, $i = (0,1)$ { i is an ordered pair $(0,1)$ }

Let us compute i^2 using Def 1.4:

$$i^2 = (0,1)(0,1) = (0\cdot 0 - 1\cdot 1,\ 0\cdot 1 + 1\cdot 0) = (-1,0) = -1$$

It is easy to check the usual rules for arithmetic, namely the axioms of a field

Prop 1.1: If $z_1, z_2, z_3, z \in \mathbb{C}$, then

1) Closure: $z_1 + z_2,\ z_1 \cdot z_2 \in \mathbb{C}$
2) Commutativity: $z_1 + z_2 = z_2 + z_1$, $z_1 \cdot z_2 = z_2 \cdot z_1$
3) Associativity: $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$; $z_1(z_2z_3) = (z_1z_2)z_3$
4) Distributivity: $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$
5) Zero and identity: $z + 0 = z$; $z \cdot 1 = z$
6) Negative and inverse: $z + (-z) = 0$, where for $z = x + iy$ $-z = -x - iy$; if $z \neq 0$, then $z \cdot z^{-1} = 1$, where $z^{-1} = \frac{1}{z} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$

Pf: let us prove the Inverse. As for the rest: check at home (all parts follow from the analogous properties of real numbers)

$$z \cdot z^{-1} = (x+iy)\left(\frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}\right) = \frac{x^2}{x^2+y^2} - i^2\frac{y^2}{x^2+y^2} + i\left(\frac{yx}{x^2+y^2} - \frac{xy}{x^2+y^2}\right)$$
$$= \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + i \cdot 0 = 1$$

We will see why the inverse is defined in exactly this way.

Def 1.5: Let $z = x + iy \in \mathbb{C}$. x is called the Real part of z and denoted by $x = \operatorname{Re} z$; y is called the Imaginary part of z, denoted by $y = \operatorname{Im} z$

The algebraic form $x+iy$ to write complex numbers allows to add and to multiply complex numbers as polynomials, but remember that $i^2 = -1$. For example:

$$(x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + ix_1y_2 + iy_1x_2 + \underset{i^2 = -1!}{i^2} y_1y_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$$

Ex 1.1: Let $z_1 = 1+i$, $z_2 = 2-i$. Compute z_1+z_2, $z_1 \cdot z_2$, z_2^{-1}

Sol: $z_1+z_2 = (1+i)+(2-i) = (1+2)+i(1+(-1)) = 3$

$z_1 \cdot z_2 = (1+i)(2-i) = (1\cdot 2 + i(-i)) + i(1(-1)+1\cdot 2) = 3+i$

z_2^{-1}: $\operatorname{Re} z_2 = x = 2$ $\operatorname{Im} z_2 = y = -1$ $\Rightarrow z_2^{-1} = \frac{2}{2^2+(-1)^2} - i\frac{-1}{2^2+(-1)^2} = \frac{2}{5} + i\frac{1}{5}$

Def 1.6: The complex conjugate of the complex number $z = x+iy$ is the complex number given by $\bar{z} = x-iy$ {bar denotes the operation of complex conjugation}

CHECK: 1) $\overline{(\bar{z})} = z$

2) $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$

3) $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

4) $\operatorname{Re} z = \frac{1}{2}(z+\bar{z}) = \frac{1}{2}(x+iy+x-iy) = x$

5) $\operatorname{Im} z = \frac{1}{2i}(z-\bar{z}) = \frac{1}{2i}(x+iy-x-iy) = y$

Using addition and multiplication we can define subtraction and division.

Def 1.7: Subtraction: $z_1 - z_2$ is a complex number z_3 such that $z_1 = z_2 + z_3$.

Using this definition we obtain: $z_3 = (x_1-x_2) + i(y_1-y_2)$

Def 1.8: Division: Let $z_2 \neq 0$. $\frac{z_1}{z_2}$ is a complex number z_3 such that $z_3 \cdot z_2 = z_1$

Let us calculate z_3. From Def 1.8 for $z_1 = x_1+iy_1$, $z_2 = x_2+iy_2$, $z_3 = x_3+iy_3$ we get:

$z_1 = x_1+iy_1 = (x_2+iy_2)(x_3+iy_3) = (x_2x_3 - y_2y_3) + i(y_2x_3 - x_2y_3) = z_2z_3$

Using Def 1.2 we have the following system of equations:

$$\begin{cases} \operatorname{Re} z_1 = x_1 = x_2x_3 - y_2y_3 = \operatorname{Re}(z_2z_3) \\ \operatorname{Im} z_1 = y_1 = y_2x_3 + x_2y_3 = \operatorname{Im}(z_2z_3) \end{cases}$$

Let us solve this system: the unknowns are x_3, y_3:

Solve and get (CHECK!) $x_3 = \frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}$ $y_3 = \frac{x_2y_1 - x_1y_2}{x_2^2+y_2^2}$

$\Rightarrow z_3 = \frac{z_1}{z_2} = \frac{x_1x_2+y_1y_2}{x_2^2+y_2^2} + i\frac{x_2y_1-x_1y_2}{x_2^2+y_2^2}$

In particular, we obtain for $\frac{1}{z}$: $x_1 = 1$, $y_1 = 0$, $x_2 = x$, $y_2 = y$

$\Rightarrow \frac{1}{z} = \frac{1\cdot x + 0\cdot y}{x^2+y^2} + i\frac{x\cdot 0 - 1\cdot y}{x^2+y^2} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$

Note that in denominator we have $z_2 \cdot \bar{z}_2 = (x_2+iy_2)(x_2-iy_2) = x_2^2+y_2^2$

Therefore, the recepie for division is: multiply the numerator and the denominator by the complex conjugate of the denominator and separate the real and imaginary parts

Ex 1.2: Calculate $z = \frac{3+4i}{1-i}$

Sol:

$$z = \frac{3+4i}{1-i} \underset{\overline{1-i} = 1+i}{=} \frac{3+4i}{1-i} \cdot \frac{1+i}{1+i} = \frac{3+3i+4i+4i^2}{1^2+(-1)^2} \underset{4i^2 = -4}{=} \frac{-1+7i}{2} = -\frac{1}{2} + i\,\frac{7}{2}$$

Ex 1.3: Prove that for any $z \in \mathbb{C}$ $\frac{1}{z} + \frac{1}{\bar{z}} \in \mathbb{R}$

Pf:

$$\frac{1}{z} + \frac{1}{\bar{z}} \underset{\overline{(\bar{z})} = z}{=} \frac{1}{z} \cdot \frac{\bar{z}}{\bar{z}} + \frac{1}{\bar{z}} \cdot \frac{z}{z} = \frac{\bar{z}}{z \cdot \bar{z}} + \frac{z}{z \cdot \bar{z}} = \frac{\bar{z}+z}{z\bar{z}} \underset{\bar{z}+z = 2\operatorname{Re} z}{=} \frac{2\operatorname{Re} z}{z\bar{z}} =$$

$$= \frac{2x}{x^2+y^2} \in \mathbb{R}$$

$\blacksquare$

Def 1.9: The modulus of $z = x+iy$: $|z| = \sqrt{x^2+y^2}$

Note: $|z| \geq 0$ always and $|z| = 0 \Longleftrightarrow z = 0 \Leftrightarrow x = y = 0$

CHECK: $|z| \geq \operatorname{Re} z$ and $|z| \geq \operatorname{Im} z$

Observe: $|z| = \sqrt{x^2+y^2} \Rightarrow |z| = \sqrt{z \cdot \bar{z}} \Rightarrow |z|^2 = z \cdot \bar{z} \Rightarrow$ if $z \neq 0$, then $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$

In the following proposition we list and prove the properties of the modulus of a complex number.

Prop 1.2: Let $z, w \in \mathbb{C}$. Then

1) $|zw| = |z||w|$
2) $|z+w| \leq |z| + |w|$ [Triangle inequality]
3) $|z-w| \geq \big||z| - |w|\big|$ [Reverse Triangle inequality]

Pf: 1) Observe that

$$|zw|^2 = (zw)\overline{(zw)} \underset{\text{for any } z_1, z_2\ \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2}{=} (zw)(\bar{z} \cdot \bar{w}) \overset{\text{(commutativity)}}{=} (z\bar{z})(w\bar{w}) = |z|^2 |w|^2$$

This proves 1) after taking square roots.

2) By Def 1.9

$$|z+w|^2 = (z+w)\overline{(z+w)} \underset{\text{for any } z_1, z_2\ \overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2}{=} (z+w)(\bar{z}+\bar{w}) = z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w}$$

$$= |z|^2 + |w|^2 + z\bar{w} + \underset{\overline{(z\bar{w})} = \bar{z}\bar{\bar{w}} = \bar{z}w = w\bar{z}}{\overline{(z\bar{w})}} \underset{\text{for any } z:\ z+\bar{z} = 2\operatorname{Re} z}{=} |z|^2 + |w|^2 + 2\operatorname{Re}(z\bar{w}) \underset{\operatorname{Re} z \leq |z|}{\leq}$$

$$\leq |z|^2 + |w|^2 + 2|z\bar{w}| \underset{|z_1 z_2| = |z_1||z_2|}{=} |z|^2 + |w|^2 + 2|z||\bar{w}| \underset{\text{for any } z\ |\bar{z}| = |z|}{=} |z|^2 + |w|^2 + 2|z||w| =$$

$$= (|z| + |w|)^2$$

Taking the square roots of both sides gives us 2)

3) We employ the trick of adding zero and apply 2):

$$|z| = |(z-w)+w| \underset{2)}{\leq} |z-w| + |w| \Rightarrow |z-w| \geq |z| - |w| \quad (a)$$

Similarly: $|z-w| = |w-z| \geq |w| - |z| = -(|z|-|w|)$ (b)

Combining (a) and (b) we get 3). ☒

Ex 1.4: Prove that if $|z_1| = |z_2| = 1$, then $\frac{z_1+z_2}{1+z_1 z_2} \in \mathbb{R}$

Pf: First, multiply and divide by the complex conjugate of the denominator

Denominator: $\overline{1+z_1 z_2} = \bar{1} + \overline{z_1 z_2} = 1 + \bar{z}_1 \bar{z}_2$

$$\Rightarrow \frac{z_1+z_2}{1+z_1 z_2} = \frac{z_1+z_2}{1+z_1 z_2} \cdot \frac{1+\bar{z}_1\bar{z}_2}{1+\bar{z}_1\bar{z}_2} = \frac{z_1+z_2+z_1\bar{z}_1\bar{z}_2+\bar{z}_1 z_2\bar{z}_2}{|1+z_1 z_2|^2} =$$

$$= \frac{z_1+z_2+|z_1|^2\bar{z}_2+\bar{z}_1|z_2|^2}{|1+z_1 z_2|^2} \underset{|z_1|=|z_2|=1}{=} \frac{z_1+z_2+\bar{z}_2+\bar{z}_1}{|1+z_1 z_2|^2} = \frac{(z_1+\bar{z}_1)+(z_2+\bar{z}_2)}{|1+z_1 z_2|^2}$$

$$= \frac{2\operatorname{Re} z_1 + 2\operatorname{Re} z_2}{|1+z_1 z_2|^2} \in \mathbb{R}$$ ☒

Ex 1.5: Prove that $\frac{z-1}{z+1}$ is purely imaginary iff $|z|=1$

Pf {Purely imaginary, namely its real part is equal to 0}

Denominator: $\overline{z+1} = \bar{z} + \bar{1} = \bar{z} + 1$

$$\Rightarrow \frac{z-1}{z+1} = \frac{z-1}{z+1} \cdot \frac{\bar{z}+1}{\bar{z}+1} = \frac{z\bar{z} - \bar{z} + z - 1}{|z+1|^2} = \frac{|z|^2 - 1 + (z - \bar{z})}{|z+1|^2}$$

$$= \frac{|z|^2 - 1 + 2i \operatorname{Im} z}{|z+1|^2}$$

Note that $|z|^2 \in \mathbb{R}$, $|z+1|^2 \in \mathbb{R}$, and $\operatorname{Im} z \in \mathbb{R}$

$$\Rightarrow \operatorname{Re}\frac{z-1}{z+1} = \frac{|z|^2-1}{|z+1|^2} \qquad \operatorname{Im}\frac{z-1}{z+1} = \frac{2\operatorname{Im} z}{|z+1|^2}$$

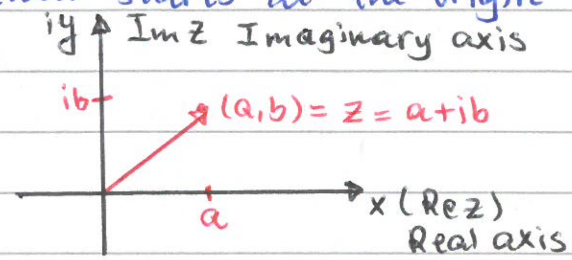
$\operatorname{Re}\frac{z-1}{z+1} = 0 \Leftrightarrow |z|^2 - 1 = 0 \Leftrightarrow |z|^2 = 1 \Leftrightarrow |z| = 1$ ☒

Lectures 2 + 3

Geometric representation of complex numbers

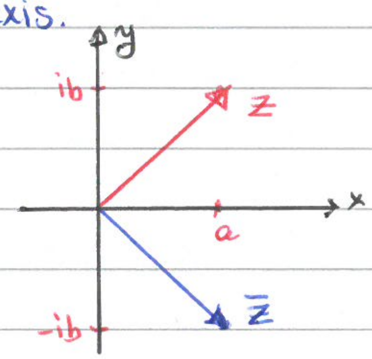
Modulus and Argument. Polar form.

Any complex number $z = a + ib$ can be numerically described as a point in the plane with the coordinates a, b or as a vector that starts at the origin $(0,0)$ and ends at the point (a,b)

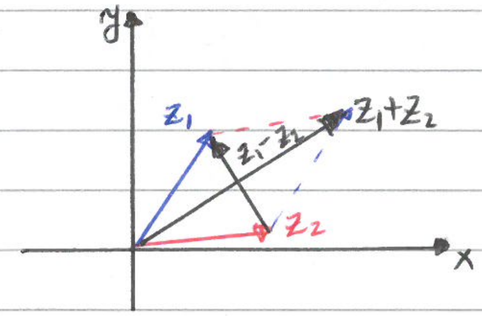


Point in the plane $\mathbb{R}^2$ – (a,b)
OR – vector connecting $(0,0)$ and (a,b)

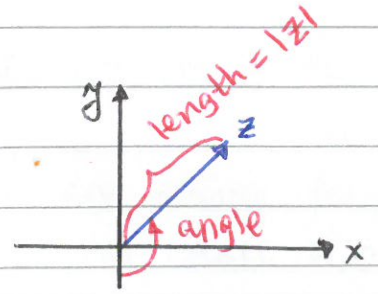
The plane (x,y) is called the complex plane. The x-axis is called the real axis and the y-axis is called the imaginary axis.



The complex conjugate of z in this plane is described by a vector that is symmetric with respect to the real axis.



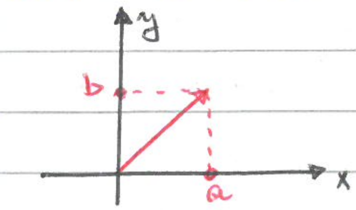
The sum of 2 complex numbers can be described as a sum of 2 vectors that start at the origin: the diagonal of the parallelogram that is constructed out of these 2 vectors. The difference between 2 complex numbers is the second (shorter) diagonal of the same parallelogram



Recall: any vector is uniquely determined by its length and by the angle that it creates with the x-asis [in the counterclockwise – positive – direction]

Def 1.9': The distance from the origin $(0,0)$ to the point $z = (x, y)$ is called the modedus of the complex number z.

Let us assume that this is the only definition and obtain Def 1.9 from Def 1.9'



By Pythagoras Theorem we get

$$|z| = \sqrt{a^2+b^2} \ (= |a+ib|)$$

This is exactly Def 1.9

Ex 2.1: Compute $|z|$ for $z_1 = 7+9i$ $z_2 = 4-3i$.

Sol: $|z_1| = |7+9i| = \sqrt{7^2+9^2} = \sqrt{130}$

$|z_2| = |4-3i| = \sqrt{4^2+(-3)^2} = \sqrt{25} = 5$

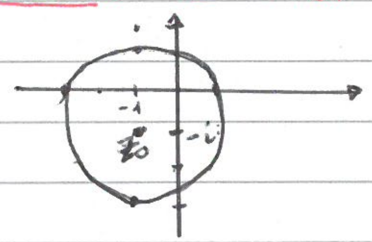
CHECK: (very easy!) The distance between any two points $z=(x,y)$ and $z_0=(x_0,y_0)$ is:

$$|z-z_0| = |(x-x_0)+i(y-y_0)| = \sqrt{(x-x_0)^2+(y-y_0)^2}$$

The equations $|z_1| = \sqrt{130}$, $|z_2| = 5$ describe the geometric place of all the points z that are at the distance $\sqrt{130}$ or 5 from the origin – namely, it is a circle. In general we have:

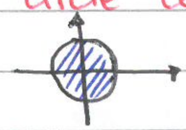
Circle with radius R centered at z_0: $|z-z_0| = R$

Ex 2.2: $|z+1+i| = 2$: circle centered at $z_0 = -(1+i)$ of radius 2



Indeed, all the points for which this equation hold lie on the circle described by $\sqrt{(x+1)^2+(y+1)^2} = 2$

The equation $|z-z_0| < R$ describes all the points that lie inside the circle centered at z_0 of radius R (but not on the circle!)

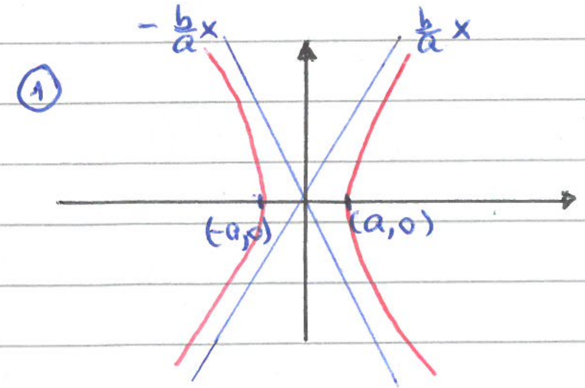


Ex 2.3: Find the geometric place of the points z for which

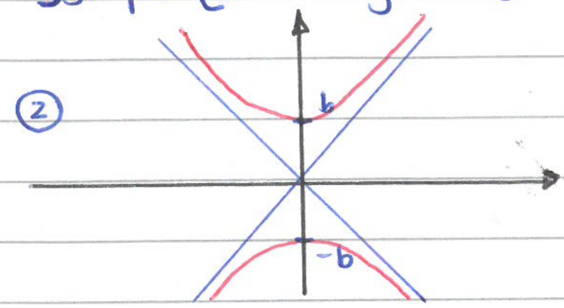
$$|z+2| - |z-2| = 1.$$

We are looking for the geometric place of all points z such that the difference of distances from the point z to two points: $z=2$ and $z=-2$ is 1. Let us show that this is hyperbola.

Recall: Hyperbola equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ – hyperbola with vertices at $(a,0)$ and $(-a,0)$ with asymptotes at $y = \pm \frac{b}{a}x$



For $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$: the vertices are at $(0,b)$ and $(0,-b)$ and assymptotes at $y=\pm\frac{a}{b}x$!



Here we have:

$|z+2|-|z-2|=1 \Rightarrow |x+iy+2|-|x+iy-2|=1 \Rightarrow |(x+2)+iy|-$
$-|(x-2)+iy|=1$

$\Rightarrow \sqrt{(x+2)^2+y^2}-\sqrt{(x-2)^2+y^2}=1$

$\Rightarrow \sqrt{(x+2)^2+y^2}=1+\sqrt{(x-2)^2+y^2}$

$\Leftrightarrow (x+2)^2+y^2 = 1+(x-2)^2+y^2+2\sqrt{(x-2)^2+y^2}$

$\Leftrightarrow (x+2)^2-(x-2)^2-1 = 2\sqrt{(x-2)^2+y^2}$

$\Leftrightarrow x^2+4x+4-x^2-4+4x-1 = 2\sqrt{(x-2)^2+y^2}$

$\Leftrightarrow 8x-1 = 2\sqrt{(x-2)^2+y^2} \Leftrightarrow (8x-1)^2 = 4\left((x-2)^2+y^2\right)$

$\Leftrightarrow 64x^2-16x+1 = 4x^2-16x+16+4y^2 \Leftrightarrow 60x^2-4y^2=15$

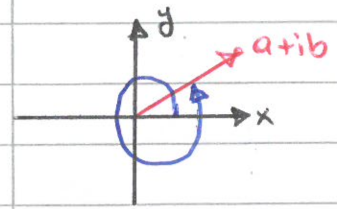
$\Leftrightarrow 4x^2-\frac{4y^2}{15}=1 \Leftrightarrow \frac{x^2}{\frac{1}{4}}-\frac{y^2}{\frac{15}{4}}=1 \Leftrightarrow \frac{x^2}{\left(\frac{1}{2}\right)^2}-\frac{y^2}{\left(\frac{\sqrt{15}}{2}\right)^2}=1$

Thus, this is hyperbola with vertices at $(\frac{1}{2},0)$ and $(-\frac{1}{2},0)$ with asymptotes $y=\pm\frac{\frac{\sqrt{15}}{2}}{\frac{1}{2}}x=\pm\sqrt{15}x$ (as in the picture 1)

Now we define the second object needed for a unique definition of a vector - an angle. This angle is called an argument of a complex number.

Def 1.10: An argument of $z\neq 0$, denoted by $\arg z$, is an angle θ such that $\cos\theta=\frac{\operatorname{Re} z}{|z|}$, $\sin\theta=\frac{\operatorname{Im} z}{|z|}$

It is only defined up to the addition of multiples of 2π! We say that θ is the principle value of the argument if $0\leq\theta<2\pi$ and denote the principal value of the argument z by $\operatorname{Arg} z$.



The argument is exactly the angle that the vector z creates with the x-axis when is measured following the positive (anti-clockwise) direction. As we said: the argument takes infinitely many values that differ one from another by a multiple of 2π.

So, now we have 3 ways to represent a complex number: as an ordered pair, an algebraic form $x+iy$, or via its modulus and an argument.

Representation: $z=(x,y)$, $z=x+iy$, or:
$x=|z|\cos\theta$, $y=|z|\sin\theta \Rightarrow z=|z|(\cos\theta+i\sin\theta) \rightarrow$ polar form
$\{(r,\theta)$- coordinates$\}$

The last form is called the polar form of a complex number.
It is more convenient to use the algebraic form to add complex numbers and it is easier to multiply using the polar form.
Let us see how we pass from an algebraic to polar representation.

Ex 2.4: Write the following numbers in polar form:
a) $-2+2i$ b) 2 c) $-i$

Sol: a) $z=-2+2i \Rightarrow |z|=\sqrt{(-2)^2+2^2}=2\sqrt{2}$, $\cos\theta=\frac{-2}{2\sqrt{2}}=-\frac{1}{\sqrt{2}}$
$\sin\theta=\frac{2}{2\sqrt{2}}=\frac{1}{\sqrt{2}} \Rightarrow \theta=\frac{3\pi}{4}\ (+2\pi k,\ k\in\mathbb{Z})$
$\Rightarrow z=2\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$
b) $z=2 \Rightarrow |z|=\sqrt{2^2+0^2}=2$; $\cos\theta=\frac{2}{2}=1$, $\sin\theta=\frac{0}{2}=0$
$\Rightarrow \theta=0\ (+2\pi k,\ k\in\mathbb{Z})$ and $z=2(\cos 0+i\sin 0)$
c) $z=-i \Rightarrow |z|=\sqrt{0^2+(-1)^2}=1$ $\cos\theta=\frac{0}{1}=0$, $\sin\theta=\frac{-1}{1}=-1$
$\Rightarrow \theta=\frac{3\pi}{2}\ (+2\pi k,\ k\in\mathbb{Z}) \Rightarrow z=1\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)$

In all previous examples the argument is up to the addition of $2\pi k$.

Rmk: $a\equiv b \pmod{c}$: means that there exists an integer $k\in\mathbb{Z}$ s.t. $a-b=kc$.
In particular, if $c=2\pi$, then $a\equiv b \pmod{2\pi}$ means that there exists $k\in\mathbb{Z}$ s.t. $a-b=2\pi k$

Important: The argument of $z=0$ is not defined!
We have already studied some of the properties of the

modulus in Prop 1.2. Three more properties (CHECK!)

Prop 1.2½: For any $z_1, z_2 \in \mathbb{C}$ $(z_2 \neq 0)$

1) $|z_1| = |\bar{z}_1|$ 2) $|z_1|^2 = z_1\bar{z}_1$ 3) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

Here we present the proof of 1) via polar coordinates; check the rest, including the ones from Prop 1.2 (in the same way)

Pf: 1) $z = |z|(\cos\theta + i\sin\theta)$ $\bar{z} = |z|(\cos\theta - i\sin\theta)$

$|z| = |z|\,|\cos\theta + i\sin\theta| = |z|\sqrt{\cos^2\theta + \sin^2\theta} = |z|$

$|\bar{z}| = |z|\,|\cos\theta - i\sin\theta| = |z|\sqrt{\cos^2\theta + (-\sin\theta)^2} = |z|$ ∎

Let us study the properties of the argument.

Prop 1.3: For any $z_1, z_2 \in \mathbb{C}$ $(z_1, z_2 \neq 0)$

1) $\arg(z_1 z_2) = \arg z_1 + \arg z_2$ 2) $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

Here we prove 1). Prove 2) in the same way.

Pf: 1) Write z_1, z_2 in polar form: let $r_1 = |z_1|$, $r_2 = |z_2|$, $\varphi_1 = \arg z_1$, $\varphi_2 = \arg z_2$. Then

$$z_{1/2} = r_{1/2}(\cos\varphi_{1/2} + i\sin\varphi_{1/2})$$

(*) $z_1 z_2 = r_1 r_2 (\cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2 + i(\sin\varphi_1\cos\varphi_2 + \sin\varphi_2\cos\varphi_1))$

$= r_1 r_2 (\cos(\varphi_1+\varphi_2) + i\sin(\varphi_1+\varphi_2))$

Since $\arg z_1 = \varphi_1 + 2\pi k$, $\arg z_2 = \varphi_2 + 2\pi m$ $\Rightarrow$

$\arg(z_1 z_2) = \varphi_1 + \varphi_2 + 2\pi n$, where $n = m+k$ $\Rightarrow$ 1) ∎

2) EXERCISE

$\Rightarrow$ The Geometric rule for multiplication: multiply moduli, add arguments.

From rule: If $z_1 = z_2 = z$, from (*): $z^2 = r^2(\cos 2\varphi + i\sin 2\varphi)$

It is easy to prove by induction on n:

De Moivre formula $z^n = |z|^n(\cos(n\varphi) + i\sin(n\varphi))$

Ex 2.5: Express $\cos 3\alpha$ and $\sin 3\alpha$ in terms of $\cos\alpha, \sin\alpha$.

Sol: Construct complex number $z = \cos\alpha + i\sin\alpha$. By De Moivre formula $z^3 = \cos 3\alpha + i\sin 3\alpha$. On the other hand:

$z^3 = (\cos\alpha + i\sin\alpha)^3 = \cos^3\alpha + 3i\cos^2\alpha\sin\alpha + 3i^2\cos\alpha\sin^2\alpha + i^3\sin^3\alpha$

$= \cos^3\alpha - 3\cos\alpha\sin^2\alpha + i(3\cos^2\alpha\sin\alpha - \sin^3\alpha)$

By Def 1.2: $z_1 = z_2 \Leftrightarrow \operatorname{Re} z_1 = \operatorname{Re} z_2$ and $\operatorname{Im} z_1 = \operatorname{Im} z_2$

Let us compare these two expressions:

$\cos 3\alpha + i\sin 3\alpha = \cos^3\alpha - 3\cos\alpha\sin^2\alpha + i(3\cos^2\alpha\sin\alpha - \sin^3\alpha)$

$\Leftrightarrow \cos 3\alpha = \cos^3\alpha - 3\cos\alpha\sin^2\alpha \qquad \sin 3\alpha = 3\cos^2\alpha\sin\alpha - \sin^3\alpha$

Ex 2.6: Define $\frac{1}{z}$ geometrically.

Sol: Let us compute the modulus and the argument.

By Prop 1.2 $\frac{1}{2}$ part 3): $\left|\frac{1}{z}\right| = \frac{1}{|z|}$ and since $z \cdot \frac{1}{z} = 1$

$0 = \arg 1 = \arg(z\frac{1}{z}) \underset{\text{Prop 1.3}}{=} \arg z + \arg\frac{1}{z} \Rightarrow \arg\frac{1}{z} = -\arg z$

Def 1.11: n-th root of a complex number z is a complex number w such that $z = w^n$ (Notation: $\sqrt[n]{z} = w$)

Ex 2.7: Find the formula for computing roots of a given $z \in \mathbb{C}$

Sol: Let $z = w^n$, let us write z and w in polar form:

$z = r(\cos\varphi + i\sin\varphi) \qquad w = \rho(\cos\theta + i\sin\theta)$

Now we express ρ and θ in terms of r and φ {r, φ are given!}

From Def 1.11: $z = w^n$ or $r(\cos\varphi + i\sin\varphi) = \rho^n(\cos(n\theta) + i\sin(n\theta))$

$\Rightarrow \rho^n = r, \ \cos(n\theta) = \cos\varphi, \ \sin(n\theta) = \sin\varphi \Rightarrow \rho = \sqrt[n]{r}, \ \theta = \frac{\varphi + 2\pi k}{n}$

for $0 \le k \le n-1$ {It is easy to see (check!) that for $k > n-1$ we will get again the same values}. Thus, we obtain

$\Rightarrow$ (**) $w = \sqrt[n]{r}\left(\cos\left(\frac{\varphi+2\pi k}{n}\right) + i\sin\left(\frac{\varphi+2\pi k}{n}\right)\right) \quad k = 0, 1, 2, \dots, n-1$

Cor 1.1 n-th root of a complex number takes exactly n different values that are placed on the vertices of regular polygon with n sides inscribed in a circle centered at 0 of radius $|z|^{\frac{1}{n}}$

Recall: regular polygon is a polygon whose sides and angles are equal

Ex 2.8: Compute $\sqrt[4]{1-i}$

Sol: We need to compute the modulus and the argument of $1-i$

$|1-i| = \sqrt{1^2 + (-1)^2} = \sqrt{2}, \ \cos\theta = \frac{1}{\sqrt{2}}, \ \sin\theta = \frac{-1}{\sqrt{2}} \Rightarrow \theta = \frac{7\pi}{4} + 2\pi k$

(or $\theta = -\frac{\pi}{4} + 2\pi k$)

$\Rightarrow 1-i = \sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$ - the polar form of $1-i$.

Using formula (**) we get

$w = \sqrt[4]{1-i} = 2^{1/8}\left(\cos\left(\frac{\frac{7\pi}{4}+2\pi k}{4}\right) + i\sin\left(\frac{\frac{7\pi}{4}+2\pi k}{4}\right)\right) \quad k = 0, 1, 2, 3$

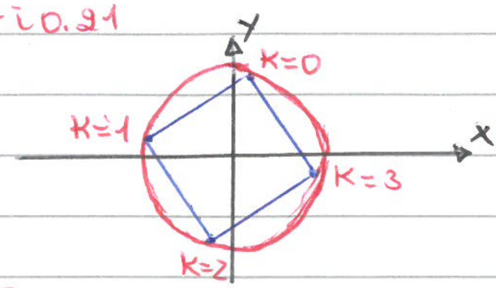
$\Rightarrow k=0 \qquad 2^{1/8}\left(\cos\frac{7\pi}{16} + i\sin\frac{7\pi}{16}\right) \approx 0.21 + i\cdot 1.07$

$k=1 \qquad 2^{1/8}\left(\cos\frac{15\pi}{16} + i\sin\frac{15\pi}{16}\right) \approx -1.07 + i\cdot 0.21$

$K=2 \quad 2^{1/8}\left(\cos\frac{23\pi}{16}+i\sin\frac{23\pi}{16}\right)\approx -0.21-i\,1.07$

$K=3 \quad 2^{1/8}\left(\cos\frac{31\pi}{16}+i\sin\frac{31\pi}{16}\right)\approx 1.07-i\,0.21$

let us draw these roots on a circle, centered at O of radius $2^{1/8}$. Connecting the neighboring roots we obtain a square



Ex 2.9: Compute all the roots of $z=\sqrt[4]{-4}$

Sol: $|-4|=4 \quad \cos\theta=\frac{-4}{4}=-1,\ \sin\theta=\frac{0}{4}=0 \Rightarrow \theta=\pi+2\pi k$

$\Rightarrow -4=4(\cos\pi+i\sin\pi)$. From (**) we get

$$\sqrt[4]{-4}=4^{1/4}\left(\cos\frac{\pi+2\pi k}{4}+i\sin\frac{\pi+2\pi k}{4}\right)=\sqrt{2}\left(\cos\left(\frac{\pi}{4}+\frac{\pi k}{2}\right)+i\sin\left(\frac{\pi}{4}+\frac{\pi k}{2}\right)\right)$$

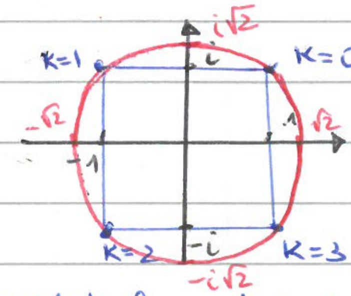
for $K=0,1,2,3$.

$K=0 \quad \sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=\sqrt{2}\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)=1+i$

$K=1 \quad \sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)=\sqrt{2}\left(-\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)=-1+i$

$K=2 \quad \sqrt{2}\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)=\sqrt{2}\left(-\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)=-1-i$

$K=3 \quad \sqrt{2}\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)=\sqrt{2}\left(\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)=1-i$



The following formula is one of the most celebrated formulae in mathematics

Thm 1.1 [Euler's formula; Euler 1760] Let $\theta\in\mathbb{R}$. Then

$$\cos\theta+i\sin\theta=e^{i\theta}$$

where $e^{i\theta}$ denotes the sum of the power series

$$e^{i\theta}=1+\frac{i\theta}{1!}+\frac{(i\theta)^2}{2!}+\dots=\sum_{n=0}^{\infty}\frac{(i\theta)^n}{n!}$$

Set $\theta=\pi$. From this formula: $e^{i\pi}=1$ – we have connected $-1, e, i, \pi$ in one formula.

We cannot prove this theorem now: first, we will need to examine the properties of the power series, however, we shall use it as a convenient notation even before we proved it. From the formula:

$\mathrm{Re}\, e^{i\theta}=1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\dots$ = The Taylor series for $\cos\theta$ (about $\theta=0$)

$\mathrm{Im}\, e^{i\theta}=\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\dots$ = The Taylor series for $\sin\theta$ (about $\theta=0$)

This formula provides a way of conversion between Cartesian coordinates $x+iy$ and the polar form (modulus and argument) as follows:

If $z\neq 0$: $|z|=r$, $\arg z=\theta \Rightarrow z=re^{i\theta}$ (the polar form)

Then $\bar{z} = x - iy = r(\cos\theta - i\sin\theta) = re^{-i\theta}$

The following should be true:

$e^{i\theta} = e^{i(\theta + 2\pi n)}$ for every $n \in \mathbb{Z}$ (Euler's formula)

$e^{i\theta} e^{i\varphi} = e^{i(\theta+\varphi)}$

$\frac{1}{e^{i\theta}} = e^{-i\theta}$

$(e^{i\theta})^n = e^{in\theta}$ for every $n \in \mathbb{Z}$

These facts are easily proved using Thm 1.1, but they are not obvious, since $e^{i\theta}$ means $1 + \frac{i\theta}{1} + \frac{(i\theta)^2}{2!} + \dots$ and raising the real number e to the purely imaginary power $i\theta$ is not well-defined at the moment.

e