Complex Variables 2006-2007

Calculus I and II revision class

Exercise 1: $x^2 + y^2 \leq 3 + 2x$

$$\Leftrightarrow x^2 - 2x + y^2 \leq 3$$
$$\Leftrightarrow (x-1)^2 - 1 + y^2 \leq 3$$
$$\Leftrightarrow (x-1)^2 + y^2 \leq 4,$$

so set is the interior of a circle centred at $(1,0)$ with radius 2.

Exercise 2: This a geometric series of the form $\sum_{n=0}^{\infty} a^n$ with $|a|<1$.

Since in this case $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$ we have

$$\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n = \frac{1}{1-\frac{3}{4}} = 4.$$

Exercise 3: (a) Since $\frac{3}{2n-1} \geq \frac{3}{2}\cdot\frac{1}{n}$ $\forall n \geq 1$, $\sum_{n=1}^{\infty} \frac{3}{2n-1} \geq \sum_{n=1}^{\infty} \frac{3}{2}\frac{1}{n} \longrightarrow \infty$,

by the divergence of the harmonic series.

Thus $\sum_{n=1}^{\infty} \frac{3}{2n-1}$ diverges.

(b) Since $\frac{n^2+2}{7-n^2} \longrightarrow -1$ as $n \to \infty$, series diverges

(c) Since $\left|\frac{(n+1)2^{-(n+1)}}{n\,2^{-n}}\right| = \frac{1}{2}\frac{n+1}{n} \longrightarrow \frac{1}{2}$ as $n \to \infty$, series converges

by ratio test.

Exercise 4 Since $\sin x$ is differentiable for $\forall x \in \mathbb{R}$, and

$1+\cos x$ " " " $\forall x \in \mathbb{R}$, and

$1+\cos x \neq 0$ wherever $x \notin \{\pi + 2k\pi \mid k \in \mathbb{Z}\}$,

f is differentiable wherever $x \neq (2k+1)\pi$ for some $k \in \mathbb{Z}$.

In this case

$$f'(x) = \frac{\cos x\,(1+\cos x) + \sin^2 x}{(1+\cos x)^2} = \frac{1+\cos x}{(1+\cos x)^2} = \frac{1}{1+\cos x}$$

Exercise 5. Since $f(x) = \cos x$, we have $f'(x) = -\sin x$, $f''(x) = -\cos x$

So $f(\frac{\pi}{3}) = \cos\frac{\pi}{3} = \frac{1}{2}\sqrt{2}$, $f'(\frac{\pi}{3}) = -\sin\frac{\pi}{3} = -\frac{1}{2}\sqrt{3}$, $f''(\frac{\pi}{3}) = -\frac{1}{2}\sqrt{2}$.

Thus first three terms of Taylor series about $\frac{\pi}{3}$ are

$$\sum_{n=0}^{2} \frac{1}{n!} f^{(n)}\left(\frac{\pi}{3}\right)\left(x-\frac{\pi}{3}\right)^n = \frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{3}\left(x-\frac{\pi}{3}\right) - \frac{1}{4}\sqrt{2}\left(x-\frac{\pi}{3}\right)^2$$

Exercise 6: (a) Let $y = \alpha x$ with $|\alpha| \neq 1$.

Then $f(x,y) = \dfrac{x^2-\alpha^2x^2}{(x+\alpha x)^2} = \dfrac{1-\alpha^2}{(1+\alpha)^2} = \dfrac{1-\alpha}{1+\alpha} \neq 0$, so

f approaches different non-zero limits as (x,y) approaches 0 along different rays, so f is not jointly continuous at 0.

(b) Use polar coordinates $x = r\cos\theta$, $y = r\sin\theta$.

Then $g(x,y) = \dfrac{r^3\cos\theta\sin^2\theta}{r^2} = r\cos\theta\sin^2\theta \to 0$ as $r \to 0$,

So g is jointly continuous at 0.

Exercise 7. $f(x,y) = e^x\cos(x+y)$.

Then $\dfrac{\partial f}{\partial x} = e^x\cos(x+y) + e^x(-\sin(x+y)) = e^x(\cos(x+y) - \sin(x+y))$

$\dfrac{\partial f}{\partial y} = e^x(-\sin(x+y)) = -e^x\sin(x+y)$.