School of Mathematical Sciences
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Examiner: Prof. O. Jenkinson

MTH6107 Chaos & Fractals
MID-TERM TEST - SOLUTIONS

Date: 8th December 2023 *Time:* 10.15am

Complete the following information:

Name	
Student Number (9 digit code)	

The test has SEVEN questions. You should attempt ALL questions. Write your calculations and answers in the space provided. Cross out any work you do not wish to be marked.

Question	Marks
1	
2	
3	
4	
5	
6	
7	
Total Marks	

Nothing on this page will be marked!

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Question 1.

Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^3$.

(a) Determine all fixed points of f.

(b) Determine, with justification, whether each fixed point is attracting or repelling.

(c) Determine the basin of attraction of each attracting fixed point.

[*15 marks*]

Answer 1.

(a) Fixed points are 0, 1, and -1, i.e. the roots of $f(x) - x = x^3 - x = x(x-1)(x+1)$.

(b) 0 is attracting, and both 1 and -1 are repelling.
Justification: $f'(x) = 3x^2$, so $|f'(0)| = 0 < 1$, while $|f'(1)| = |f'(-1)| = 3 > 1$.

(c) The basin of attraction of 0 is $(-1, 1)$.
A justification for this (which wasn't asked for in this question) is that $f^n(x) = x^{3^n}$, so $\lim_{n\to\infty} f^n(x) = 0$ if and only if $|x| < 1$.

Answer 1. (*Continue*)

Question 2.

Suppose $f : [0,1] \to [0,1]$ is defined by

$$f(x) = \begin{cases} 1-2x & \text{if } x \in [0,1/2) \\ x - 1/2 & \text{if } x \in [1/2,1]\,. \end{cases}$$

(a) Sketch the graph of f.

(b) Determine the orbit under f of the point $1/2$.

(c) Determine the set of $n \in \mathbb{N}$ such that f has an n-cycle.

[*15 marks*]

Answer 2.

(a) Full marks for any reasonable sketch showing f as continuous and piecewise linear.

(b) The orbit of $1/2$ is $\{1/2, 0, 1\}$, i.e. an orbit of least period 3.

(c) Th required set is $\mathbb{N}$, i.e. f has an n-cycle for every $n \in \mathbb{N}$.
The justification for this (which wasn't asked for in this question) is that f is continuous, and has a 3-cycle (by (b) above), so Sharkovskii's Theorem implies the result.

Answer 2. (*Continue*)

Question 3.
Suppose $f : [0,1] \to [0,1]$ is defined by $f(x) = \frac{1}{2}\sin(\pi x)$.

(a) Sketch the graph of f.

(b) Determine the fixed points of f.

(c) For each fixed point of f, determine whether it is attracting or repelling, being careful to justify your answer.

[*15 marks*]

Answer 3.
(a) Full marks for any reasonable sketch (showing f as a non-negative function on $[0,1]$, increasing on the sub-interval $[0,1/2]$ and decreasing on the sub-interval $[1/2,1]$).

(b) 0 and $1/2$ are the only fixed points.

(c) 0 is repelling, and $1/2$ is attracting.
Justification: $f'(x) = \frac{\pi}{2}\cos(\pi x)$, so $|f'(0)| = \pi/2 > 1$, and $|f'(1/2)| = 0 < 1$.

Answer 3. (*Continue*)

Question 4.

Suppose $f : [0,1] \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 10x & \text{if } x \in [0,1/2) \\ 10x - 9 & \text{if } x \in [1/2,1] \,. \end{cases}$$

(a) Sketch the graph of f.

(b) What does it mean to say that a point $x \in [0,1]$ is *non-escaping*?

(c) Give an explicit example of a rational number $x \in (0,1)$ that is non-escaping, expressing x both in decimal form and in the form p/q for $p, q \in \mathbb{N}$.

[*15 marks*]

Answer 4.

(a) Full marks for any reasonable sketch showing f as linear with slope 10 on $[0,1/2)$, rising from value 0 to value 5, then a discontinuity at the point $1/2$, and f linear with slope 10 on $[1/2,1]$, rising from value -4 to value 1.

(b) It means that $f^n(x) \in [0,1]$ for all $n \geq 0$.

(c) One example is the point $x = 0.09090909\ldots = 1/11$.

More generally, the set of non-escaping points consists of precisely those real numbers in $[0,1]$ with decimal expansion containing only the digits 0 and 9, and if this expansion is chosen to be periodic (or eventually periodic) then the corresponding number x will be rational.

Answer 4. (*Continue*)

Question 5.

Suppose $\phi_1 : [0,1] \to [0,1]$ and $\phi_2 : [0,1] \to [0,1]$ are given by $\phi_1(x) = (x+1)/5$ and $\phi_2(x) = (x+3)/5$, and define Φ by $\Phi(A) = \phi_1(A) \cup \phi_2(A)$.

(a) Write down an explicit expression for the set $\Phi([0,1])$.

(b) Write down an explicit expression for the set $\Phi^2([0,1])$.

[*10 marks*]

Answer 5.

(a) $\Phi([0,1]) = [1/5, 2/5] \cup [3/5, 4/5]$.

(b) $\Phi^2([0,1]) = [6/25, 7/25] \cup [8/25, 9/25] \cup [16/25, 17/25] \cup [18/25, 19/25]$.

Answer 5. (*Continue*)

Question 6.
Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = -x + 1$.

(a) Is f topologically conjugate to the map $g_1 : \mathbb{R} \to \mathbb{R}$ defined by $g_1(x) = x + 1$? Justify your answer.

(b) Is f topologically conjugate to the map $g_2 : \mathbb{R} \to \mathbb{R}$ defined by $g_2(x) = -x - 1$? Justify your answer.

[*20 marks*]

Answer 6.
(a) No, f is not topologically conjugate to g_1.
Justification: Note that g_1 has no fixed points, but f has a fixed point (at $1/2$).

(b) Yes, f is topologically conjugate to g_2.
Justification: Assume $h(x) = ax + b$ for some (as yet undetermined) constants a and b satisfies $h(f(x)) = g_2(h(x))$, i.e. $a(-x + 1) + b = -(ax + b) - 1$, i.e. $-ax + a + b = -ax - b - 1$, so the constant a can be arbitrary (though should be non-zero to ensure h is a homeomorphism), and b must satisfy $a + b = -b - 1$, i.e. $b = (-1 - a)/2$.
So for example we could choose $a = -1$ and $b = 0$, so $h(x) = -x$, and we then verify that $h(f(x)) = -(-x + 1) = x - 1 = -(-x) - 1 = -h(x) - 1 = g_2(h(x))$, as required.

Answer 6. (*Continue*)

Question 7.

Let $D : [0,1) \to [0,1)$ be the doubling map, defined by $D(x) = 2x \pmod 1$, and for all $x \in [0,1)$ let $b(x) = b_1 b_2 b_3 b_4 \ldots$, where $b_i = 0$ if $D^{i-1}(x) \in [0, 1/2)$ and $b_i = 1$ if $D^{i-1}(x) \in [1/2, 1)$.

(a) If $x = \pi/4$, what are the first three terms b_1, b_2, b_3 of the sequence $b(x) = b_1 b_2 b_3 b_4 \ldots$?

(b) Determine the point $x \in [0,1)$ with periodic sequence $b(x) = \overline{00001} = 00001\,00001\,00001 \ldots$.

[*10 marks*]

Answer 7.

(a) $b_1 b_2 b_3 = 110$.

The justification (which wasn't required in this question) is that $x = \pi/4 \in (3/4, 1)$, since $\pi \in (3,4)$, so $b_1 = 1$.

Then $D(x) = 2x - 1 \in (1/2, 1)$, so $b_2 = 1$.

Then $D^2(x) = 4x \pmod 1 = \pi \pmod 1 = 0.141\ldots \in [0, 1/2)$, so $b_3 = 0$.

(b) $x = 1/31$ (i.e. the sum of the geometric series $\sum_{i=1}^{\infty} 1/2^{5i} = \frac{1/32}{1-1/32}$).

Answer 7. (*Continue*)