MTH6112 Actuarial Financial Engineering
Coursework Week 2

1. Suppose that the odds of m possible outcomes of an experiment are $o_i > 0$, where $i = 1, \dots, m$. In other words, the return function is given by

$$r_i(j) = \begin{cases} o_i & \text{if } j = i; \\ -1 & \text{if } j \neq i. \end{cases}$$

Use the Arbitrage Theorem to show that either

$$\sum_{i=1}^{m} (1 + o_i)^{-1} = 1\,,$$

or there is an arbitrage opportunity.

Solution

By the Arbitrage Theorem it suffices to determine under what conditions there exist positive real numbers $p_1, \dots, p_m$ with $p_1 + \cdots + p_m = 1$ such that

$$\sum_{j=1}^{m} p_j r_i(j) = 0 \quad (\forall i = 1, \dots, m)\,.$$

Now, for fixed i, the above sum becomes

$$\sum_{j=1}^{m} p_j r_i(j) = p_i o_i + \sum_{\substack{j=1 \\ j \neq i}}^{m} p_j \cdot (-1) = p_i o_i - \sum_{\substack{j=1 \\ j \neq i}}^{m} p_j$$

$$= p_i o_i - \sum_{j=1}^{m} p_j + p_i = p_i o_i - (1 - p_i) = p_i(o_i + 1) - 1\,,$$

that is,

$$p_i(o_i + 1) - 1 = 0\,,$$

hence

$$p_i = \frac{1}{1 + o_i}\,.$$

Note that

$$0 \leq \frac{1}{1 + o_i} \leq 1\,,$$

and the Arbitrage Theorem now implies that, either these probabilities satisfy $p_1 + \cdots + p_m = 1$, i.e.

$$\sum_{i=1}^{m} \frac{1}{1+o_i} = 1\,,$$

or there is an arbitrage opportunity.

2. Let $S(j)$ be a 2-period Binomial model with parameters S, u, d, r and suppose that $u > 1+r > d$. Prove that then the risk-neutral probability is given by

$$\mathbb{P}((1,0)) = \mathbb{P}((0,1)) = pq, \quad \mathbb{P}((1,1)) = p^2, \quad \mathbb{P}((0,0)) = q^2,$$

where $p = \frac{1+r-d}{u-d}$, $q = 1-p$.

Hints First of all, introduce the following notation: $p_{ij} = \mathbb{P}((i,j))$.

One possibility to prove the above is to consider the following two investment strategies.

Strategy 1: buy 1 share at time 0 and sell it at time 1. For each outcome (i,j), compute the return function $r_1(i,j)$. Now, use the no-arbitrage equations to prove that $p_{1,0} + p_{1,1} = p$ (and hence $p_{0,1} + p_{0,0} = q$.

Strategy 2: if $S(1) = Su$ then buy 1 share at time 1 and sell it at time 2. If $S(1) = Sd$ then don't buy anything (and hence don't earn or lose money).

Once again, for each outcome (i,j), compute the return function $r_2(i,j)$. Now, use the no-arbitrage equations and observe that they imply that $p_{1,1} = cp$ and $p_{1,0} = cq$ for some c. The rest now follows easily.

Solution Consider Strategy 1. Then the value of return at time $t = 0$ (the present value of return) is

$$r_1(1,1) = \frac{Su}{1+r} - S, \;\; r_1(1,0) = \frac{Su}{1+r} - S, \;\; r_1(0,1) = \frac{Sd}{1+r} - S, \;\; r_1(0,0) = \frac{Sd}{1+r} - S.$$

Note that here $r_1(1,1) = r_1(1,0)$ and $r_1(0,1) = r_1(0,0)$ since the gain does not depend on the behaviour of the price after time 1. By the Arbitrage Theorem, the expectation of return should be 0. So we have

$$\mathbb{E}(r_1) = \sum_{i,j} r_1(i,j) p_{i,j} = p_{1,1}(\frac{Su}{1+r} - S) + p_{1,0}(\frac{Su}{1+r} - S) + p_{0,1}(\frac{Sd}{1+r} - S) + p_{0,0}(\frac{Sd}{1+r} - S)$$
$$= (p_{1,1} + p_{1,0})(\frac{Su}{1+r} - S) + (p_{0,1} + p_{0,0})(\frac{Sd}{1+r} - S) = 0.$$

Denote $\tilde{p} = p_{1,1} + p_{1,0}$, $\tilde{q} = p_{0,1} + p_{0,0}$. Then

$$\tilde{p}(\frac{u}{1+r} - 1) + \tilde{q}(\frac{d}{1+r} - 1) = 0 \quad \text{and} \quad \tilde{p} + \tilde{q} = 1. \tag{1}$$

Solving this system of two equations, we obtain $\tilde{p} = \frac{1+r-d}{u-d} = p$ and $\tilde{q} = 1-p$.

Consider Strategy 2. Then at time $t = 1$, the values of r_2 are

$$r_2(1,1) = \frac{Su^2}{1+r} - Su, \quad r_2(1,0) = \frac{Sud}{1+r} - Su, \quad r_2(0,1) = r_2(0,0) = 0.$$

Once again, by the Arbitrage Theorem, the expectation of return should be 0. So we have

$$\mathbb{E}(r_2) = \sum_{i,j} r_2(i,j)p_{i,j} = p_{1,1}(\frac{Su^2}{1+r} - Su) + p_{1,0}(\frac{Sud}{1+r} - Su) + p_{0,1} \times 0 + p_{0,0} \times 0 = 0.$$

This equation can be re-written as $p_{1,1}(\frac{u}{1+r} - 1) + p_{1,0}(\frac{d}{1+r} - 1) = 0$ and we see that it is the first equation from (1). Hence there is a c such that $p_{1,1} = cp$ and $p_{1,0} = cq$. Since we also know that $p_{1,1} + p_{1,0} = p$, we obtain that $c = p$ and therefore $p_{1,1} = p^2$, $p_{1,0} = pq$.

Remark If you don't understand the last 3 lines of the argument, then just solve the system of two equations: $p_{1,1}(\frac{u}{1+r} - 1) + p_{1,0}(\frac{d}{1+r} - 1) = 0$ and $p_{1,1} + p_{1,0} = p$.

3. The price of a share follows a 3-period Binomial model $S(j)$ with parameters S, u, d, r and suppose that $u > 1 + r > d$. A derivative on this share operates according to the following rules. If $S(2) \geq Sud$ then the owner of the derivative can buy the share at time 2 for K, where $K < Sud$. If $S(2) < Sud$ then the owner of the derivative can sell the share at time 3 for K_1, where $Su^2d \geq K_1 > Sud^2$. What is the no-arbitrage price C of this derivative?

   Solution As explained in lecture, it is convenient to describe the sequence of prices $(S(1), S(2), S(3))$ in terms of sequences of 0 (zeros) and 1 (ones). E. g., the sequence of prices (Su, Sud, Su^2d) is described by the sequence (101) (where 1 corresponds to the price going up by the factor u and 0 corresponds to the price going down by the factor d). There are 8 different sequences (outcomes of the experiment) and we shall compute the return provided by the derivative for each outcome at time 0.

We start with sequences satisfying the condition $S(2) \geq Sud$ These are sequences $(111),\ (110),\ (101),\ (100),\ (011),\ (010)$.

In the first two of the 6 listed outcomes, the price of the share at time 2 is $S(2) = su^2$). Therefore the payoff at time 2 is $Su^2 - K$ (because you buy the share for K and sell it for its market price Su^2). The value of this payoff at time 0 is $(1+r)^{-2}(Su^2 - K)$ and hence the pure gain (return) computed at time 0 is

$$R(111) = R(110) = (1+r)^{-2}(Su^2 - K) - C,$$

where C is the price one pays for the derivative at time 0.

Similarly, in the other 4 cases, we have

$$R(101) = R(100) = R(011) = R(010) = (1+r)^{-2}(Sud - K) - C.$$

In the remaining 2 cases, (001) and (000), the price of the share at time 2 is $Sd^2 < Sud$ and therefore the payoff time is 3. If the outcome is (001) then the payoff occurs at time 3 and is $K_1 - Sud^2$ (which is the difference between the price at which the the owner is allowed to sell the share and the price of the share). The value of this payoff at time 0 is $(1+r)^{-3}(K_1 - Sud^2)$ and the pure gain at time 0 is

$$R(001) = (1+r)^{-3}(K_1 - Sud^2) - C.$$

Similarly,

$$R(000) = (1+r)^{-3}(K_1 - Sd^3) - C.$$

By the Arbitrage Theorem, the expectation of return over the risk-neutral probability on the space of outcomes is 0:

$$\tilde{\mathbb{E}}(R) = \sum_{i_1,i_2,i_3} R(i_1 i_2 i_3)\tilde{\mathbb{P}}(i_1 i_2 i_3) = 0$$

We know the risk-neutral probabilities for this model, namely $\tilde{\mathbb{P}}(i_1 i_2 i_3) = p^{i_1+i_2+i_3} q^{3-(i_1+i_2+i_3)}$, where $p = \frac{1+r-d}{u-d}$, $q = \frac{u-1-r}{u-d}$. Substituting the computed values of return into this formula, we obtain:

$$[(1+r)^{-2}(Su^2 - K) - C](p^3 + p^2 q) + [(1+r)^{-2}(Sud - K) - C](2p^2 q + 2pq^2)$$
$$+ [(1+r)^{-3}(K_1 - Sud^2) - C]pq^2 + [1+r)^{-3}(K_1 - Sd^3) - C]q^3 = 0.$$

We can simplify this relation by observing that $p^3 + p^2 q = p^2(p+q) = p^2$, and $2p^2 q + 2pq^2 = 2pq$. Since $p^2 + 2pq + pq^2 + q^3 = 1$, we have

$$p^2(1+r)^{-2}(Su^2 - K) + 2pq(1+r)^{-2}(Sud - K)$$
$$+ pq^2(1+r)^{-3}(K_1 - Sud^2) + q^3(1+r)^{-3}(K_1 - Sd^3) - C = 0$$

and we obtain:

$$\begin{aligned} C =& p^2(1+r)^{-2}(Su^2 - K) + 2pq(1+r)^{-2}(Sud - K) \\ &+ pq^2(1+r)^{-3}(K_1 - Sud^2) + q^3(1+r)^{-3}(K_1 - Sd^3). \end{aligned}$$