Actuarial Financial Engineering

Week 2

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Introduction

We apply the **no-arbitrage** principle in order to value derivative contracts.

The basic idea is that if we can construct a **portfolio** that **replicates** the payoff from the derivative under every possible circumstance, then that portfolio must have **the same value** as the derivative.
So, by valuing the replicating portfolio we can value the derivative

Introduction

The models discussed at the beginning of this topic represent the underlying **share price** as a **stochastic process** in **discrete time** and with a **discrete state space** – in fact a geometric random walk.

Later on, we will discuss the **continuous-time** and **continuous-state space** analogue, geometric Brownian motion (or the lognormal model),
which can be interpreted as the **limiting case** of the binomial model as the size of the time steps tends to zero.

Overview of this week

2.1. Assumptions of One-period Binomial model

Here we will consider a model for stock prices in **discrete time**.
The model will seem very **simple and naïve** at first sight, but:

- it introduces the **key concepts** of financial economic pricing and
- it leads us to the celebrated **Black-Scholes model as a limiting case**.

2.1. Assumptions of One-period Binomial model

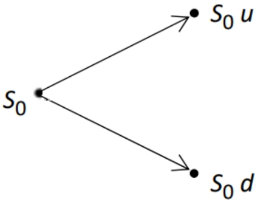
In the binomial model, it is assumed that:

- there are no trading costs
- there are no taxes
- there are no minimum or maximum units of trading
- stock and bonds can only be bought and sold at discrete times 1, 2, ...
- the principle of no arbitrage applies.

2.2. Basic structure of One-Period Binomial model

Definition 2.1

Consider a model of asset prices $S(i)$, where $i = 0, 1$ (only 1 time-period), such that today's price $S(0) = S$ is given, while next period's asset price is either $S(1) = S\,u$ or $S(1) = S\,d$.
Here u and d are positive real numbers satisfying $0 < d < u$ and let r be the nominal interest rate per time period.
This model is called **the one-period binomial model** (also called the one-step binomial model).



2.2. Basic structure of One-Period Binomial model

Question 1: Under what condition on u, d, r, there is no arbitrage opportunity in the binomial model?

2.2. Basic structure of One-Period Binomial model

To answer Question 1, we need review the Arbitrage Theorem:

Theorem 2.1 (The Arbitrage Theorem)

Given a set of bets $r_1, \ldots, r_n$ one and only one of the following statements holds:

1. *There exists a probability vector $\mathbf{p} = (p_1, \ldots, p_m)$ such that $p_1 > 0, \ldots, p_m > 0$ and such that for each $i = 1, \ldots, n$*

$$p_1 r_i(1) + p_2 r_i(2) + \cdots + p_m r_i(m) = 0. \tag{1}$$

2. *There is an arbitrage.*

2.2. Basic structure of One-Period Binomial model

Answer to Question 1:
By definition, there are exactly 2 possible outcomes of this experiment, namely either $S(1) = Sd$ or $S(1) = Su$.
Accordingly, equation (1) from the Arbitrage Theorem leads to

$$p_1\, r_1(1) + p_2\, r_1(2) = 0, \tag{2}$$

where $r_1(j)$ is the return function with $j = 1$ corresponding to the case $S(1) = Su$ and $j = 2$ corresponding to the case $S(1) = Sd$.

2.2. Basic structure of One-Period Binomial model

Answer to Question 1 (cont.):

We can now compute the return function.

Since the (discounted) value of $S(1)$ at time 0 is $S(1)/(1+r)$, therefore the return (the gain) computed at time 0 is

$$\text{return at time } 0 = \frac{S(1)}{1+r} - S\,.$$

Thus

$$r_1(1) = \underbrace{\frac{Su}{1+r} - S}_{\substack{\text{present value of gain}\\ \text{if asset price moves up}}} \quad \text{and } r_1(2) = \underbrace{\frac{Sd}{1+r} - S}_{\substack{\text{present value of gain}\\ \text{if asset price moves down}}}$$

and equation (2) now becomes

$$p_1\left(\frac{Su}{1+r} - S\right) + p_2\left(\frac{Sd}{1+r} - S\right) = 0\,.$$

2.2. Basic structure of One-Period Binomial model

Answer to Question 1 (cont.):

Since $p_2 = 1 - p_1$, we have

$$\frac{p_1 u}{1+r} + \frac{(1-p_1)d}{1+r} - 1 = 0\,,$$

so

$$p_1 = \frac{1+r-d}{u-d}, \quad p_2 = \frac{u-1-r}{u-d}\,.$$

Now, the condition $p_1 \geq 0$ implies $1 + r - d \geq 0$ or, equivalently, $1 + r \geq d$.
Similarly, $p_2 \geq 0$ implies $u \geq 1 + r$.
We thus see that in this model the necessary and sufficient condition to have no arbitrage is $d \leq 1 + r \leq u$.
In fact, the borderline cases, that when $u = 1 + r$ or $d = 1 + r$, are not important (explain why).

2.2. Basic structure of One-Period Binomial model

Answer to Question 1 (cont.):
Thus the condition for no arbitrage is

$$d < 1 + r < u.$$

These specific probabilities p_1 and p_2 for an upward and downward movement of the asset price, respectively, under the above condition on the parameters, are called **risk-neutral probabilities (RNP)**.

2.2. Basic structure of One-Period Binomial model

Definition 2.2 (Risk Neutral Measure)

$\sim$ is a probability measure such that each price is exactly equal to the **discounted expectation** of the price under the measure.
Such a measure exists if and only if the market is **arbitrage-free**.

2.2. Basic structure of One-Period Binomial model

Exercise:
In a One-Period Binomial Model,
assume $\hat{p}_1$ is the probability of going up under the **real-world** probability measure,
while p_1 is the probability of going up under the **risk-neutral** probability measure.
Show why we would normally find that $\hat{p}_1 > p_1$.
This is your homework.

2.2. Basic structure of One-Period Binomial model

Before moving to Question 2, recall:

Definition 2.3 (Option)

An **option** is a derivative contract which gives its **buyer** the right, but not the obligation, to buy or sell an underlying asset for a specific *strike price* K on or prior to the *maturity date* T.
The **seller** of the option is then obliged to fulfill the transaction, to sell or buy, in the scenario when the owner chooses to "exercise" the option.
The **buyer** has to pay a "premium" in order to buy this right.

Remark: The buyer of the option is also called the the *owner* of the option;
the *strike price* is the same as the *exercise price*;
the *maturity date/time* is the same as the *expiration time* of the option.

2.2. Basic structure of One-Period Binomial model

Definition 2.4 (European call option)

A **European call option** is contract which gives its buyer the right, but not the obligation, to buy the underlying asset for a **strike price** K on the **maturity date** T.

2.2. Basic structure of One-Period Binomial model

Question 2:
Consider a European call option for which the underlying asset is the one described above. What is the no-arbitrage price C of this option if its strike price is K and maturity time is 1?

2.2. Basic structure of One-Period Binomial model

Consider a corresponding call option which costs C at time 0, has maturity time 1 and strike price K.

Let $r_2(j)$ be the corresponding return function. Then

$$r_2(1) = \underbrace{\frac{(Su-K)^+}{1+r} - C}_{\substack{\text{present value of gain} \\ \text{if asset price moves up}}} \quad \text{and} \quad r_2(2) = \underbrace{\frac{(Sd-K)^+}{1+r} - C}_{\substack{\text{present value of gain} \\ \text{if asset price moves down}}}$$

and hence

$$p_1\, r_2(1) + p_2\, r_2(2) = p_1 \left(\frac{(Su-K)^+}{1+r} - C\right) + p_2 \left(\frac{(Sd-K)^+}{1+r} - C\right) = 0.$$

0 $\qquad\qquad\qquad\qquad$ 1

$-C$ $\qquad\qquad\qquad\qquad$ $(Su-K)^+, p_1$

$\qquad\qquad\qquad\qquad\qquad$ $(Sd-K)^+, p_2$

2.2. Basic structure of One-Period Binomial model

Solving this equation we obtain

$$C = p_1 \frac{(Su-K)^+}{1+r} + p_2 \frac{(Sd-K)^+}{1+r}$$

For example, if the parameters of this model satisfy $Sd < K < Su$, then the European call option price is

$$C = p_1 \frac{(Su-K)^+}{1+r} = \frac{(1+r-d)(Su-K)}{(1+r)(u-d)}\,.$$

0 $\qquad\qquad\qquad\qquad\qquad\qquad$ 1

$-C$

$(Su-K)^+, p_1$

$(Sd-K)^+, p_2$

2.3. One-Period Binomial Model: derivatives & hedging

What is a derivative on a share?

Definition 2.5 (Derivative)

A $\sim$ on a share (or, more generally, on an asset) is an agreement according to which one pays £C for entering into this agreement and, after a certain period of time, is paid the amount of money which depends on the *outcome of the experiment*.

There are only 2 possible outcomes of the experiment, and every derivative can be described by 3 numbers:

- the price C which is paid by the buyer of the derivative at time 0,
- and the payoff function taking values V_1 and V_2.
  Say, V_1 is paid if the price s of the share becomes su, and V_2 is paid if the price s of the share becomes sd.

2.3. One-Period Binomial Model: derivatives & hedging

Question. What should be the no-arbitrage price of this derivative?
Answer. The price C is given by

$$C = \frac{1}{1+r} p_1 V_1 + \frac{1}{1+r} p_2 V_2 = \frac{1+r-d}{(1+r)(u-d)} V_1 + \frac{u-1-r}{(1+r)(u-d)} V_2,$$

where $p_1,\ p_2$ are the risk-neutral probabilities derived above.

2.3. One-Period Binomial Model: derivatives & hedging

Proof. Let $r_2(j)$ be the return function for this derivative computed at time 1. Then

$$r_2(1) = V_1 - C(1+r), \quad r_2(2) = V_2 - C(1+r).$$

Explanation: we borrow C from the bank at time 0, buy the derivative, and pay back $C(1+r)$ to the bank at time 1.

If the price of the asset becomes su, we are paid V_1 and so $r_2(1)$ is our pure gain (or return).

If the price of the asset becomes sd, we are paid V_2 and $r_2(2)$ is our pure gain.

0	1
	Payoff$= V_1, p_1$
Buy the derivative $-C$	
	Payoff$= V_2, p_2$
Borrow from bank C	Return bank $-C(1+r)$

2.3. One-Period Binomial Model: derivatives & hedging

Proof (cont.). Next, by the Arbitrage Theorem,

$$p_1 r_2(1) + p_2 r_2(2) = 0 \quad \text{or, equivalently,} \quad p_1 V_1 + p_2 V_2 - (p_1 + p_2) C(1 + r) = 0.$$

Rearranging and taking into account that $p_1 + p_2 = 1$, we obtain the result. $\square$

Remark. In the particular case of the call option, the payoff function is given by $V_1 = (su - K)^+, \ V_2 = (sd - K)^+$.

2.3. One-Period Binomial Model: derivatives & hedging

Statement of the problem.
A trader sells for £C a derivative with the payoff function (V_1, V_2) described above. Can he invest his money at time 0 so that to be able to meet his obligation at time 1?

Being in the framework of the One-Period Binomial Model means that the trader is allowed to do only two things:

- buy shares
- deposit money into the bank.

2.3. One-Period Binomial Model: derivatives & hedging

Question. How many shares should he buy and how much money should he deposit into the bank in order to be able to meet his obligation (no matter what the cost of the share would be at time 1)?
Answer. Let x be the number of shares the trader should buy and y be the amount of money he should deposit into the bank. Then

$$x = \frac{V_1 - V_2}{s(u-d)}, \quad y = \frac{uV_2 - dV_1}{(1+r)(u-d)}. \tag{3}$$

2.3. One-Period Binomial Model: derivatives & hedging

Proof.

Investment 1:

	0	1
		xsu
Invest in share	$-x \cdot s$	
		xsd
Deposit in bank	$-y$	$y(1+r)$

Investment 2:

	0	1
		V_1
Invest in derivative	$-C$	
		V_2

2.3. One-Period Binomial Model: derivatives & hedging

Proof (cont.). If the trader buys x shares and deposits y into the bank at time 0, then at time 1 his capital will be either $xsu + y(1+r)$ if the price goes up or $xsd + y(1+r)$ if the price goes down. Hence, we get two equations:

$$\begin{cases} xsu + y(1+r) = V_1 \\ xsd + y(1+r) = V_2 \end{cases}$$

Solving these equations gives the result. (Remark that if these two equations are satisfied then the trader gains the money he needs to meet his obligation, no matter what the outcome is.)

2.3. One-Period Binomial Model: derivatives & hedging

Proof (cont.). Note now that the capital P the trader has to invest at time 0 is

$$P = xs + y = \frac{V_1 - V_2}{s(u-d)}s + \frac{uV_2 - dV_1}{(1+r)(u-d)} = (1+r)^{-1}\left(\frac{1+r-d}{u-d}V_1 + \frac{u-1-r}{u-d}V_2\right) \tag{4}$$

$$= (1+r)^{-1}(p_1 V_1 + p_2 V_2). \tag{5}$$

Comparing this expression with the one for the price C of the derivative obtained above, we see that $P = C$. Thus, selling the derivative for C guarantees the trader the possibility to meet his obligation!

2.3. One-Period Binomial Model: derivatives & hedging

Remark The equality $P = C$ (the capital $=$ the price of the derivative) could be predicted. Indeed, we have two investments.
Investment 1: buy the derivative for C.
Investment 2: buy $\frac{V_1 - V_2}{s(u-d)}$ shares and deposit $\frac{uV_2 - dV_1}{(1+r)(u-d)}$ into the bank.
Both investments produce the same result at time 1. Hence, by the Law of One Price, their cost should be the same.

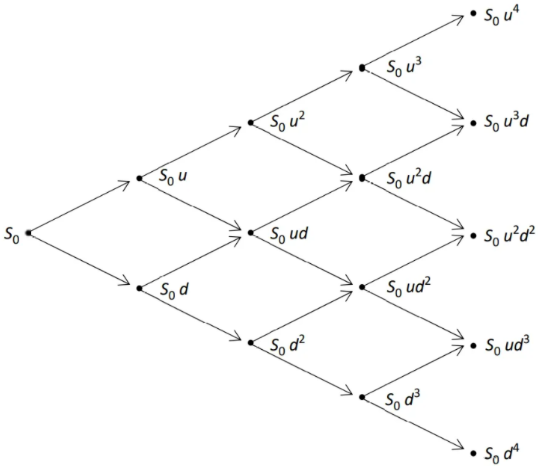
3. Multiperiod Binomial model

3. Multiperiod Binomial model

Definition 3.1 (Multiperiod Binomial model)

Consider a model of share prices $S(t)$, where $t = 0, 1, \ldots, n$, such that, given $S(t-1)$, then we have either $S(t) = u\,S(t-1)$ or $S(t) = d\,S(t-1)$, for $1 \leq i \leq n$, where $S(0) = S$ is given.
Here u and d are positive real numbers with $0 < d < u$. Let r be the nominal interest rate per time period.
This model is called the **Multiperiod Binomial model**.

Multiperiod Binomial model



3. Multiperiod Binomial model

We shall always suppose that $d < 1 + r < u$ holds, implying that there is no-arbitrage in the market (see the one-period model).
The above implies that at time

- $t = 1$ the possible values of the share price are Su and Sd.
- $t = 2$ the possible values are Su^2, Sud, Sd^2.

  $\vdots$

- $t = n$ the possible values are $Su^n,\ Su^{n-1}d,\ \ldots,\ Sd^n$.

3. Multiperiod Binomial model

The experiment now consists of observing the behaviour of prices and the outcomes are described by the sequence

$$\mathbf{s} = \big(S(0), S(1), \ldots, S(n)\big). \tag{6}$$

Define another sequence of numbers

$$\mathbf{i} = (i_1, \ldots, i_n) \text{ such that for } 1 \leq j \leq n \;,\; i_j = \begin{cases} 1 \;, & \text{if } S(j) = u\, S(j-1) \\ 0 \;, & \text{if } S(j) = d\, S(j-1) \end{cases} . \tag{7}$$

Clearly, there is a one-to-one correspondence between sequences $\mathbf{s}$ in (6) and $\mathbf{i}$ in (7). It is easy to see (explain this!) that for any intermediate time $k \in \{1, \ldots, n\}$, we have

$$S(k) = S\, u^{\sum_{j=1}^{k} i_j}\, d^{\,k - \sum_{j=1}^{k} i_j} .$$

3. Multiperiod Binomial model

Set, by definition,

$$\mathbb{P}(\mathbf{i}) \equiv \mathbb{P}(i_1, \dots, i_n) \stackrel{\text{def}}{=} p^{\sum_{j=1}^n i_j} (1-p)^{n-\sum_{j=1}^n i_j}, \quad \text{where} \quad p = \frac{1+r-d}{u-d}. \tag{8}$$

Probabilities of this form arise in the description of Bernoulli trials.
Let X_j be a sequence independent identically distributed random variables having the Bernoulli(p) distribution then (8) can be written as

$$\mathbb{P}(X_1 = i_1, \dots, X_n = i_n) \stackrel{\text{def}}{=} p^{\sum_{j=1}^n i_j} (1-p)^{n-\sum_{j=1}^n i_j}, \quad \text{where} \quad p = \frac{1+r-d}{u-d}. \tag{9}$$

3. Multiperiod Binomial model

Theorem 3.1 (3.1)

Consider the multiperiod binomial model with parameters S, u, d, r defined above. Then the condition $d < 1 + r < u$ is necessary and sufficient for no-arbitrage opportunities in the market. Moreover, the corresponding risk neutral probabilities are given by (8).

Proof is not examinable.

3. Multiperiod Binomial model

Example

I. Buy a share at time 0 for S and sell it at time n for $S(n)$.
Denote by $r(i_1, i_2, \dots, i_n)$ the corresponding return function.
Then Theorem 3.1 states that

$$\sum_{i_1, i_2, \dots, i_n} r(i_1, i_2 \dots, i_n)\, \mathbb{P}(X_1 = i_1, \dots, X_n = i_n) \equiv \tilde{\mathbb{E}}\big[r(X_1, X_2, \dots, X_n)\big] = 0,$$

where $\tilde{\mathbb{E}}$ is the expectation computed over the **risk-neutral probability**.
Remark. In the future, $\tilde{\mathbb{E}}$ will be used whenever it is important to distinguish between expectations over the real life probability and expectations over the risk-neutral probability.

3. Multiperiod Binomial model

Example

II. Buy a share at time 0 for S and sell it at time 3 if the price $S(3) \geq Su^2d$. On the other hand, if $S(3) < su^2d$, then sell it at time n for $S(n)$. Denote by $\tilde{r}(i_1, i_2, \ldots, i_n)$ the corresponding return function. Then again Theorem 3.1 states that

$$\sum_{i_1, i_2, \ldots, i_n} \tilde{r}(i_1, i_2, \ldots, i_n)\, \mathbb{P}(X_1 = i_1, \ldots, X_n = i_n) \equiv \tilde{\mathbb{E}}\big[\tilde{r}(X_1, X_2, \ldots, X_n)\big] = 0.$$

3. Multiperiod Binomial model

Example

II (cont.). Instead of using the random variables X_i, for $i = 1, \ldots, n$, it is convenient to introduce the random variable Y given by

$$Y = \sum_{i=1}^{n} X_i \,.$$

Note that Y has a binomial distribution, that is,

$$\mathbb{P}(Y = y) = \binom{n}{y} p^y (1-p)^{n-y} \,,$$

and that

$$S(n) = S u^Y d^{n-Y} \,.$$

3. Multiperiod Binomial model

We are now in a position to answer the equivalent to Question 2 for the multiperiod model considered in this section.

Question 3. Consider a European call option for which the underlying asset is the one described above.
What is the no-arbitrage price C of this option if its strike price is K and maturity time is $T = n$?

3. Multiperiod Binomial model

Theorem 3.2

The price C of the European call option with maturity time $T = n$ and strike price K is:

$$C = \frac{1}{(1+r)^n} \sum_{y=0}^{n} (Su^y d^{\,n-y} - K)^+ \, \mathbb{P}(Y = y) = \frac{1}{(1+r)^n} \, \tilde{\mathbb{E}}\left[(Su^Y d^{\,n-Y} - K)^+\right], \quad (10)$$

where as before the symbol $\tilde{\mathbb{E}}$ denotes the expectation of a random variable over the risk-neutral probability.

3. Multiperiod Binomial model

Proof.
If $S(n) \leq K$, then the present value of the payoff of the call option is 0.
If $S(n) > K$, then the present value of the payoff of the call option is $(S(n) - K)(1 + r)^{-n}$.
Thus, the present value of the payoff of the call option is

$$\frac{(S(n) - K)^+}{(1 + r)^n} .$$

Now, the present value of the return from buying the call option is

$$\frac{(S(n) - K)^+}{(1 + r)^n} - C .$$

3. Multiperiod Binomial model

By the Arbitrage Theorem, in order to have no-arbitrage in the market, we must have

$$\sum_{i_1, i_2, \ldots, i_n} r(i_1, i_2, \ldots, i_n)\, \mathbb{P}(X_1 = i_1, \ldots, X_n = i_n) = 0$$

$$\Rightarrow \sum_{i_1,\, i_2,\, \ldots,\, i_n} \Big[\frac{(Su^{\sum_{i=1}^n i_i} d^{\,n-\sum_{i=1}^n i_i} - K)^+}{(1+r)^n} - C\Big]\, \mathbb{P}(X_1 = i_1, \ldots, X_n = i_n) = 0$$

$$\Rightarrow \sum_{y=0}^{n} \Big[\frac{(Su^y d^{\,n-y} - K)^+}{(1+r)^n} - C\Big]\, \mathbb{P}(Y = y) = 0\,,$$

so by rearranging we get

$$C = \frac{1}{(1+r)^n} \sum_{y=0}^{n} (Su^y d^{\,n-y} - K)^+\, \mathbb{P}(Y = y) = \frac{1}{(1+r)^n}\, \tilde{\mathbb{E}}\big[(Su^Y d^{\,n-Y} - K)^+\big]\,,$$

where the symbol $\tilde{\mathbb{E}}$ denotes, as before, the expectation of a random variable over the risk-neutral probability. □