MTH6112 Actuarial Financial Engineering
Coursework Week 11

1. An analyst is using a two-state continuous-time model to study the credit risk of zero-coupon bonds issued by different companies. The risk-neutral transition intensity function is:

   - $\widetilde{\lambda}_A(s) = 0.0148$ for Company A, and
   - $\widetilde{\lambda}_B(s) = 0.01s^2$ for Company B,

   where s measures time in years from now. The analyst observes that the credit spread on a 3 year zero-coupon bond just issued by Company B is twice that on a 3-year zero-coupon bond just issued by Company A. Given that the average recovery rate in the event of default, δ, where $0 < \delta < 1$, is the same for both companies, calculate δ. What should be the relation between the recovery rates of these two companies for there not to be arbitrage in the market?

   Remark. The credit spread on a zero-coupon bond is the difference between the yield on the bond and the yield on a similar bond issued by the government. I.e., for company i it is equal to $R_i(t,T) - r$, where r is the risk-free interest rate and

   $$R_i(t,T) = -\frac{1}{T-t}\log B(t,T).$$

   The price of a zero-coupon bond in a two-state model was derived in the Lecture and is equal to

   $$B(t,T) = e^{-r(T-t)}\left(\delta + (1-\delta)\,e^{-\int_t^T \widetilde{\lambda}(s)ds}\right).$$

   Solution:

   For the spot rate curve, $R(t,T)$, one has

   $$R(t,T) = -\frac{1}{T-t}\log B(t,T).$$

   Let C_i be the credit spread on a 3 year zero-coupon bond just issued by company i. Then by the definition of the credit spread we have

$$\begin{aligned} C_i &= R_i(0,3) - r = -\frac{1}{3}\log\left[e^{-3r}\left(\delta + (1-\delta)\,e^{-\int_0^3 \widetilde{\lambda}_i(s)ds}\right)\right] - r \\ &= -\frac{1}{3}\log\left(\delta + (1-\delta)\,e^{-\int_0^3 \widetilde{\lambda}_i(s)ds}\right). \end{aligned}$$

The analyst observes that the credit spread on a bond just issued by Company B is twice that on a bond just issued by Company A, thus

$$-\frac{1}{3}\log\left(\delta + (1-\delta)\,e^{-\int_0^3 \widetilde{\lambda}_B(s)ds}\right) = -\frac{2}{3}\log\left(\delta + (1-\delta)\,e^{-\int_0^3 \widetilde{\lambda}_A(s)ds}\right). \quad (1)$$

The above is an equation in one variable δ. Let us first calculate corresponding integrals.

$$\begin{aligned} \int_0^3 \widetilde{\lambda}_A(s)\,ds &= 0.0148\int_0^3 1ds = 0.0148s\Big|_0^3 = 0.0444, \\ \int_0^3 \widetilde{\lambda}_B(s)\,ds &= 0.01\int_0^3 s^2ds = 0.01\frac{s^3}{3}\Big|_0^3 = 0.09. \end{aligned}$$

By multiplying (1) by -3 and then exponentiating one gets

$$\delta + (1-\delta)\,e^{-0.09} = \left(\delta + (1-\delta)\,e^{-0.0444}\right)^2.$$

Remark: Please bear in mind, that one needs to keep precise values in this problem. Otherwise you will get unrealistic results.

To shorten notations let us introduce $\alpha = e^{-0.0444}$ and $\beta = e^{-0.09}$. The equation can be rewritten as

$$\delta^2(1-\alpha)^2 + \delta\left(2\alpha - 2\alpha^2 + \beta - 1\right) + \alpha^2 - \beta = 0.$$

Discriminant of the above is equal to

$$D = \left(\beta - \alpha^2 - (1-\alpha)^2\right)^2 - 4\left(\alpha^2 - \beta\right)(1-\alpha)^2 = \left(\beta - \alpha^2 + (1-\alpha)^2\right)^2.$$

And the solution to equation is then given by

$$\begin{cases} \delta_1 &= \frac{1-\beta+2\alpha^2-2\alpha+\beta-\alpha^2+(1-\alpha)^2}{2(1-\alpha)^2} = 1 \\ \delta_2 &= \frac{1-\beta+2\alpha^2-2\alpha-\beta+\alpha^2-(1-\alpha)^2}{2(1-\alpha)^2} = \frac{\alpha^2-\beta}{(1-\alpha)^2}. \end{cases}$$

Remark: Solution $\delta_1 = 1$ should not be a surprise. This just confirms that in the case of no default, the credit spread is zero for any company.

Plugging now corresponding numbers, we obtain

$$\delta = \frac{e^{-0.0888} - e^{-0.09}}{(1 - e^{-0.0444})^2} = 0.5818.$$

2. This question is covered in the Slides of this week. Please dirty your hands and do it independently to check if you are able to calculate them.
   Let $\xi_i, i = 1, 2, \ldots, n$ be independent random variables taking the values ± 1 with probability $\mathbb{P}[\xi_1 = 1] = 1/2$.
   We denote by $\mathcal{F}_n$ the σ-algebra generated by $\xi_1, \xi_2, \ldots, \xi_n$. Further we denote

$$S_n = \sum_{i=1}^{n} \xi_j.$$

   Finally, let τ be a random variable taking values in $\mathbb{N}$, with

$$\mathbb{E}[\tau] < \infty,$$

   and τ being independent of all ξ_i. Compute the following conditional expectations.

   a) $\mathbb{E}[e^{\xi_1+\xi_2-\xi_3}|\xi_1, \xi_2]$
   b) $\mathbb{E}[S_n|\mathcal{F}_{n-1}]$
   c) $\mathbb{E}[S_n^2 - n|\mathcal{F}_{n-1}]$
   d) $\mathbb{E}[e^{S_n}|\mathcal{F}_{n-1}]$
   e) $\mathbb{E}[S_n^2|S_{n-1}]$
   f) $\mathbb{E}[S_\tau^2|\tau]$

   Solution:

   a) $\mathbb{E}[e^{\xi_1+\xi_2-\xi_3}|\xi_1, \xi_2]$
      In this case we deal with the conditional expectation with respect to random variables, and thus we "know the information" about their values. Saying rigorously, all functions of ξ_1, ξ_2 are measurable and we can use Property **Exp.3**. At the same time ξ_3 is independent of ξ_1, ξ_2 and we can use Property **Exp.4**. This leads to

$$\mathbb{E}\left[e^{\xi_1+\xi_2-\xi_3}|\xi_1, \xi_2\right] \overset{\textbf{Exp.3}}{=} e^{\xi_2+\xi_2}\mathbb{E}\left[e^{-\xi_3}|\xi_1, \xi_2\right] \overset{\textbf{Exp.4}}{=} e^{\xi_2+\xi_2}\mathbb{E}\left[e^{-\xi_3}\right].$$

For the last average we have

$$\mathbb{E}\left[\mathrm{e}^{-\xi_3}\right] = \mathrm{e}^{-1} \cdot \frac{1}{2} + \mathrm{e}^{+1} \cdot \frac{1}{2} = \cosh(1),$$

and thus

$$\mathbb{E}\left[\mathrm{e}^{\xi_1+\xi_2-\xi_3}|\xi_1, \xi_2\right] = \mathrm{e}^{\xi_1+\xi_2} \cosh(1).$$

b) $\mathbb{E}[S_n|\mathcal{F}_{n-1}]$

In this case we deal with the conditional expectation with respect to a σ-algebra. Thus we need to understand first what is measurable with respect to this σ-algebra and what is independent. In simple words, which variables this σ-algebra keeps an "**information**" about. It follows from the problem statement that this σ-algebra does keep an "information" about first $n-1$ variables. Therefore we can write

$$\mathbb{E}\left[S_n|\mathcal{F}_{n-1}\right] = \mathbb{E}\left[S_{n-1} + \xi_n|\mathcal{F}_{n-1}\right] \overset{\textbf{Exp.1}}{=} \mathbb{E}\left[S_{n-1}|\mathcal{F}_{n-1}\right] + \mathbb{E}\left[\xi_n|\mathcal{F}_{n-1}\right]$$
$$\overset{\textbf{Exp.3}}{=} S_{n-1} + \mathbb{E}\left[\xi_n|\mathcal{F}_{n-1}\right] \overset{\textbf{Exp.4}}{=} S_{n-1} + \mathbb{E}\left[\xi_n\right].$$

For the last average we have

$$\mathbb{E}\left[\xi_n\right] = (1) \cdot \frac{1}{2} + (-1) \cdot \frac{1}{2} = 0,$$

and thus

$$\mathbb{E}\left[S_n|\mathcal{F}_{n-1}\right] = S_{n-1}.$$

c) $\mathbb{E}[S_n^2 - n|\mathcal{F}_{n-1}]$

Analogously to the above, we calculate the expectation given an information about first $n-1$ variables. Thus,

$$\mathbb{E}\left[S_n^2 - n|\mathcal{F}_{n-1}\right] \overset{\textbf{Exp.1}}{=} \mathbb{E}\left[(S_{n-1} + \xi_n)^2|\mathcal{F}_{n-1}\right] - n$$
$$\overset{\textbf{Exp.1}}{=} \mathbb{E}\left[S_{n-1}^2|\mathcal{F}_{n-1}\right] + 2\mathbb{E}\left[S_{n-1}\xi_n|\mathcal{F}_{n-1}\right] + \mathbb{E}\left[\xi_n^2|\mathcal{F}_{n-1}\right] - n$$
$$\overset{\textbf{Exp.3}}{=} S_{n-1}^2 + 2S_{n-1}\mathbb{E}\left[\xi_n|\mathcal{F}_{n-1}\right] + \mathbb{E}\left[\xi_n^2|\mathcal{F}_{n-1}\right] - n$$
$$\overset{\textbf{Exp.4}}{=} S_{n-1}^2 + 2S_{n-1}\mathbb{E}\left[\xi_n\right] + \mathbb{E}\left[\xi_n^2\right] - n.$$

Mean value of ξ_n was calculated above and is equal to 0. For the second moment we have

$$\mathbb{E}\left[\xi_n^2\right] = 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = 1,$$

and therefore

$$\mathbb{E}\left[S_n^2 - n|\mathcal{F}_{n-1}\right] = S_{n-1}^2 - (n-1)\,.$$

Rremark. In the above two examples one can notice the same phenomena. Indeed we have shown for two different processes, namely $X_n = S_n$ and $X_n = S_n^2 - n$, and σ-algebra $\mathcal{F}_{n-1}$ the validity of following relation

$$\mathbb{E}\left[X_n|\mathcal{F}_{n-1}\right] = X_{n-1}.$$

d) $\mathbb{E}[\mathrm{e}^{S_n}|\mathcal{F}_{n-1}]$
Analogously to the above we split the exponent into two parts: one containing variables $\xi_1, \xi_2, \ldots, \xi_{n-1}$ and another containing ξ_n and then use the Properties **Exp.3**, **Exp.4**.

$$\mathbb{E}\left[\mathrm{e}^{S_n}|\mathcal{F}_{n-1}\right] = \mathbb{E}\left[\mathrm{e}^{S_{n-1}}\mathrm{e}^{\xi_n}|\mathcal{F}_{n-1}\right] \overset{\textbf{Exp.3}}{=} \mathrm{e}^{S_{n-1}}\mathbb{E}\left[\mathrm{e}^{\xi_n}|\mathcal{F}_{n-1}\right] \overset{\textbf{Exp.4}}{=} \mathrm{e}^{S_{n-1}}\mathbb{E}\left[\mathrm{e}^{\xi_n}\right] = \mathrm{e}^{S_{n-1}}\cosh(1).$$

e) $\mathbb{E}[S_n^2|S_{n-1}]$
In this case we need to emphasize that the conditional expectation with respect to S_{n-1} is not the same as the conditional expectation with respect to $\mathcal{F}_{n-1}$. Expectation of the form $\mathbb{E}\left[\cdot|S_{n-1}\right]$ is the one which is taken with respect to a σ-algebra that "keeps the information" about S_{n-1} only, but not about independent variables $\xi_1, \xi_2, \ldots, \xi_{n-1}$. But we still can successfully use the same technique to obtain

$$\begin{aligned}\mathbb{E}\left[S_n^2|S_{n-1}\right] &\overset{\textbf{Exp.1}}{=} \mathbb{E}\left[S_{n-1}^2|S_{n-1}\right] + 2\mathbb{E}\left[S_{n-1}\xi_n|S_{n-1}\right] + \mathbb{E}\left[\xi_n^2|S_{n-1}\right]\\ &\overset{\textbf{Exp.3}}{=} S_{n-1}^2 + 2S_{n-1}\mathbb{E}\left[\xi_n|S_{n-1}\right] + \mathbb{E}\left[\xi_n^2|S_{n-1}\right]\\ &\overset{\textbf{Exp.4}}{=} S_{n-1}^2 + 2S_{n-1}\mathbb{E}\left[\xi_n\right] + \mathbb{E}\left[\xi_n^2\right] = S_{n-1}^2 + 1.\end{aligned}$$

f) $\mathbb{E}[S_\tau^2|\tau]$
This is a more complicated example. Namely we have a random number of terms forming S_τ. Thus we start from the definition of a conditional expectation via conditional distribution. Because τ is independent of all ξ_n we can fix the value of τ to be equal n. Then the variable S_n has some distribution. We are interested in its second moment

$$\mathbb{E}\left[S_n^2\right] = \mathbb{E}\left[\sum_{j,k}\xi_j\xi_k\right] = \sum_j \mathbb{E}\left[\xi_j^2\right] + \sum_{j\neq k}\mathbb{E}\left[\xi_j\xi_k\right] = n + \sum_{j\neq k}\mathbb{E}\left[\xi_j\right]\mathbb{E}\left[\xi_k\right] = n.$$

This means that

$$\mathbb{E}\left[S_\tau^2|\tau = n\right] = n = \tau,$$

and thus

$$\mathbb{E}\left[S_\tau^2|\tau\right] = \tau.$$

3. This question is also covered in the Slides of Week 11. Make sure you are able to solve it.

   a) Let W_t be a standard Brownian Motion/Wiener Process and $\{\mathcal{F}_t\}_{t\geq 0}$ be a corresponding natural filtration. Let $B_t = B_0 + \mu t + \sigma W_t$ be a BM with corresponding drift and volatility. Show that W_t is a martingale.
   Solution:
   W_t is a martingale because of

$$\mathbb{E}\left[W_t|\mathcal{F}_s\right] = \mathbb{E}\left[W_t - W_s + W_s|\mathcal{F}_s\right] \overset{\textbf{Exp.1}}{=} \mathbb{E}\left[W_t - W_s|\mathcal{F}_s\right] + \mathbb{E}\left[W_s|\mathcal{F}_s\right]$$
$$\overset{\textbf{Exp.3, past}}{=} \mathbb{E}\left[W_t - W_s|\mathcal{F}_s\right] + W_s \overset{\textbf{Exp.4, future}}{=} \mathbb{E}\left[W_t - W_s\right] + W_s = W_s.$$

   But B_t is not a martingale unless μ is equal to zero. Indeed,

$$\mathbb{E}\left[B_t|\mathcal{F}_s\right] = \mathbb{E}\left[B_t - B_s + B_s|\mathcal{F}_s\right] \overset{\textbf{Exp.1}}{=} \mathbb{E}\left[B_t - B_s|\mathcal{F}_s\right] + \mathbb{E}\left[B_s|\mathcal{F}_s\right]$$
$$\overset{\textbf{Exp.3}}{=} \mathbb{E}\left[B_t - B_s|\mathcal{F}_s\right] + B_s \overset{\textbf{Exp.4}}{=} \mathbb{E}\left[B_t - B_s\right] + B_s = \mu\left(t-s\right) + B_s \neq B_s, \quad \text{for } \mu \neq 0.$$

   b) Show that under the same assumption, W_t^2 is a not martingale, but $W_t^2 - t$ is.
   Solution:

$$\mathbb{E}\left[W_t^2|\mathcal{F}_s\right] = \mathbb{E}\left[\left(W_t - W_s + W_s\right)^2|\mathcal{F}_s\right]$$
$$\overset{\textbf{Exp.1}}{=} \mathbb{E}\left[\left(W_t - W_s\right)^2|\mathcal{F}_s\right] + 2\mathbb{E}\left[W_s\left(W_t - W_s\right)|\mathcal{F}_s\right] + \mathbb{E}\left[W_s^2|\mathcal{F}_s\right]$$
$$\overset{\textbf{Exp.3}}{=} \mathbb{E}\left[\left(W_t - W_s\right)^2|\mathcal{F}_s\right] + 2W_s\mathbb{E}\left[W_t - W_s|\mathcal{F}_s\right] + W_s^2$$
$$\overset{\textbf{Exp.4}}{=} \mathbb{E}\left[\left(W_t - W_s\right)^2\right] + 2W_s\mathbb{E}\left[W_t - W_s\right] + W_s^2 = t - s + W_s^2.$$

   At the same time if one introduces X_t to be equal $W_t^2 - t$, then it follows from the above

$$\mathbb{E}\left[X_t|\mathcal{F}_s\right] = \mathbb{E}\left[W_t^2|\mathcal{F}_s\right] - t = W_s^2 - s = X_s,$$

   and thus X_s is a martingale with respect to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

c) Let $S_t = \mathrm{e}^{B_t}$ be a Geometric Brownian Motion starting at $S_0 = \mathrm{e}^{B_0}$ and having drift μ and volatility σ. Show that this process is not a martingale in general, but is a martingale for $\mu = -\frac{\sigma^2}{2}$.

Solution:

This follows from the below

$$\mathbb{E}\left[S_t|\mathcal{F}_s\right] = \mathbb{E}\left[\mathrm{e}^{\mu(t-s)+\sigma(W_t-W_s)}S_s|\mathcal{F}_s\right] \overset{\textbf{Exp.3}}{=} \mathrm{e}^{\mu(t-s)}S_s\mathbb{E}\left[\mathrm{e}^{\sigma(W_t-W_s)}|\mathcal{F}_s\right]$$
$$\overset{\textbf{Exp.4}}{=} \mathrm{e}^{\mu(t-s)}S_s\mathbb{E}\left[\mathrm{e}^{\sigma(W_t-W_s)}\right] = S_s\mathrm{e}^{\mu(t-s)+\frac{\sigma^2}{2}(t-s)} = S_s\mathrm{e}^{\left(\mu+\frac{\sigma^2}{2}\right)(t-s)}.$$