MTH6112 Actuarial Financial Engineering
Coursework Week 10

1. A company has just issued zero-coupon bonds with expiration time of 2 years and the total nominal value of £3 million. The total value of the company now stands at £4 million. A continuously compounded interest rate is 5% per annum. The total value of the company follows the Geometric Brownian motion with parameters $\mu = 0.4$ and $\sigma = 0.2$.

   a) Under the Merton model, find the current value of the shareholders' equity.

   b) In 1 year time, the company's value drops by 20%. What is the probability of the company's default on its obligation to bondholders.

   c) (optional, will not appear in the exam) After the value drops, the company decides to issue new bonds with the same maturity date. It is known that this move was the cause of the shareholders' equity drop by 70%. What is the debt of the company to bondholders at the end of the 2 years' period?

      Hint We apply Newton's method to get the numerical solution. You may use the formula for $\frac{\partial C}{\partial K}$.

   Solution:

   a) According to the Merton model, the shareholders can be treated as having a European Call option on the assets of the company with strike price $L_0 = £3$ million and maturity $T = 2$ years. Thus the value of shareholders' equity is equal to

   $$E_0 = F_0 \Phi(\omega) - L_0 e^{-rT} \Phi\left(\omega - \sigma\sqrt{T}\right).$$

   First we calculate ω by using the formula

   $$\begin{aligned} \omega &= \frac{\log \frac{F_0}{L_0} + rT}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T} = 1.5121, \\ \omega - \sigma\sqrt{T} &= \frac{\log \frac{F_0}{L_0} + rT}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T} = 1.2292. \end{aligned}$$

   Plugging the numbers into the above formula gives

   $$E_0 = £\,1.3217\,\text{millions}.$$

b) In one year the value of the company drops to $F_1 = F_0 \cdot 0.8 = £\,3.2$ millions. The company's value, under the assumptions of the Black-Scholes theory, follows the Geometric Brownian Motion.

$$F_{1+t} = F_1 e^{\mu t + \sigma W_t}.$$

The company would default if the value of the company drops below the repayment value L_0.

$$\begin{aligned}
\mathbb{P}\left(\text{default}\right) &= \mathbb{P}\left(F_2 < L_0\right) = \mathbb{P}\left(F_1 e^{\mu+\sigma W_1} < L\right) = \mathbb{P}\left(W_1 < \frac{\ln\frac{L}{F_1} - \mu}{\sigma}\right) \\
&= \Phi\left(\frac{\ln\frac{L}{F_1} - \mu}{\sigma\sqrt{T}}\right) = \Phi\left(-2.323\right) = 1 - \Phi\left(2.323\right) = 1 - 0.9898 = 0.0102.
\end{aligned}$$

c) Denote by L_1 the total nominal value of bonds issued by the company. Obviously, $L_1 > L_0$. The B-S formula provides us with the equation for L_1. Namely

$$E_1 = F_1 \Phi\left(\omega_1\right) - L_1 e^{-rT} \Phi\left(\omega_1 - \sigma\sqrt{T}\right), \tag{1}$$

where $E_1 = 0.3E_0 = £0.3965$ million, $F_1 = £3.2$ millions, $T = 1$ year and $\omega_1 = \frac{\ln\frac{F_1}{L_1} + r}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}$.
This equation can be solved numerically (but not explicitly). To do that, we first rewrite (1) as follows:

$$F_1 \Phi\left(\omega_1\right) - L_1 e^{-rT} \Phi\left(\omega_1 - \sigma\sqrt{T}\right) - E_1 = 0. \tag{2}$$

We now define a function $g(x)$ by

$$g(x) = F_1 \Phi\left(\omega(x)\right) - x e^{-r} \Phi\left(\omega(x) - \sigma\right) - E_1, \tag{3}$$

where $\omega(x) = \frac{\ln F_1 - \ln x + r}{\sigma} + \frac{1}{2}\sigma$. Note that $g(x)$ is just the left hand side of (2) with L_1 replaced by x (we don't keep the $\sqrt{T}$ in (3) and in $\omega(x)$ because $T = 1$).
It is obvious that $g(L_1) = 0$ and we thus want to solve the equation

$$g(x) = 0.$$

To do that, we use Newton's method. We first compute $g'(x)$:

$$\begin{aligned}
g'(x) &= F_1 \Phi'\left(\omega(x)\right)\omega'(x) - x e^{-r} \Phi'\left(\omega(x) - \sigma\right)\omega'(x) - e^{-r}\Phi\left(\omega(x) - \sigma\right) \\
&= \left[F_1 \phi\left(\omega(x)\right) - x e^{-r} \phi\left(\omega(x) - \sigma\right)\right]\omega'(x) - e^{-r}\Phi\left(\omega(x) - \sigma\right) \\
&= -e^{-r}\Phi\left(\omega(x) - \sigma\right) < 0. \qquad (4)
\end{aligned}$$

Here $\phi\left(\omega\right)=\frac{1}{\sqrt{2\pi}}e^{-\frac{\omega^2}{2}}$.

(**Exercise.** Check that the expression in the square brackets is really equal to zero. This derivative was in fact computed the formula for $\frac{\partial C}{\partial K}$.)

Since for $x>0$

$$g''(x)=-e^{-r}\Phi'\left(\omega(x)-\sigma\right)\omega'(x)=e^{-r}\phi\left(\omega(x)-\sigma\right)\frac{1}{x\sigma}>0,$$

the conditions under which Newton's method can be applied in the interval $(0,\infty)$ are satisfied. Next, it is natural to choose L_0 as the first approximation to L_1. So we set $x_0=L_0=3$ and compute x_1, x_2, ... using the recursion

$$x_{n+1}=x_n-\frac{g(x_n)}{g'(x_n)}\quad\text{for}\quad n=0,\ 1,\ 2,\ ... \tag{5}$$

Before we start the actual calculations, it is useful to observe that they can be simplified if we transform this expression for x_{n+1} by plugging (3) and (4) into (5). We then obtain

$$x_{n+1}=\frac{F_1\Phi(\omega(x_n))-E_1}{\Phi(\omega(x_n)-\sigma)}\times e^r. \tag{6}$$

It is also useful to note that $\omega(x)=6.166-5\ln x$.

We have $\omega(x_0)=0.673$, $\omega(x_0)-\sigma=0.473$, $\Phi(0.673)=0.7489$, $\Phi(0.473)=0.6818$ and so

$$x_1=\frac{3.2*0.7489-0.3965}{0.6818}e^{0.05}=3.08695$$

The second iteration gives $\omega(x_1)=0.53$, $\omega(x_1)-\sigma=0.33$, $\Phi(0.53)=0.7019$, $\Phi(0.33)=0.6293$ and so

$$x_2=\frac{3.2*0.7019-0.3965}{0.6293}e^{0.05}=3.08979$$

The third iteration gives $\omega(x_2)=0.525$, $\omega(x_2)-\sigma=0.325$, $\Phi(0.525)=0.70$, $\Phi(0.325)=0.6270$ and so

$$x_3=\frac{3.2*0.70-0.3965}{0.627}e^{0.05}=3.0909$$

The next iteration would already require a precision which is higher than the one we adopted above. (Check this statement by computing x_4!)

Thus, $L_1=$ £3.0909 million is the debt of the company at the end of the 2 years' time.