Actuarial Financial Engineering

Week 10

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February 3, 2023

Overview of this week

15. Credit Risk

15.1 The Merton model
15.2 Two-state intensity-based model for credit ratings
15.3 The Jarrow-Lando-Turnbull (JLT) model

15. Credit Risk

Credit risk is the risk that a person or an organisation will fail to make a payment they have promised.
There are several kinds of models addressing the problem of how to estimate credit risk.
In this module, we consider two types of models.

- The **structural models** are the models which link default events (see the definitions below) with the structure of a corporate entity's equity and debt.
  The Merton model is the simplest example of a structural model.
- The **reduced-form models**

15. Credit Risk

- The **structural models**
- The **reduced-form models** are statistical models which use observed market statistics along with the data on the default-free market to model the movement of the credit rating of the bonds issued by the corporate entity over time.
  The main output of such a model is the distribution of the time of default.
  They are called "reduced-form" because they ignore specific data concerning the company which issues the bond. Instead, they use credit ratings issued by credit rating agencies such as Standard and Poor's and Moody's.
  In turn, when setting their ratings, the credit rating agencies would use detailed data specific to the corporate entity issuing the bond.

15. Credit Risk

Let's recall/introduce some terminology which will be used below.
A *bond* (or a fixed income security) is a debt instrument created to raise capital. More precisely:

Definition 15.1 (Bond)

A **bond** is a loan agreement between a bond issuer and an investor in which the bond issuer is obligated to pay a specified amount of money at specified future dates.

Definition 15.2 (Default-free bond)

A **default-free bond** is the one which repays interest and the principal with absolute certainty.

Government bonds may be viewed as an example of default-free bonds but corporate bonds may default. **Default** may mean that the payment
1. is rescheduled, 2. is reduced, 3. is continued but at reduced rate,
4. is completely wiped out.

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Definition 15.3 (Credit event)

A **credit event** is an event that will trigger the default of a bond.

Examples of credit events are:

1. Failure to pay either the capital or a coupon.
2. Bankruptcy.
3. Rating downgrade of the bond (we shall discuss more about the ratings later).

Definition 15.4 (Recovery rate)

Recovery rate is the fraction of the default amount that can be recovered trough bankruptcy proceedings or some other form of settlement.

15.1. The Merton model

The Merton model is an example of a model describing the **structure** of the value (total capital) of a corporate entity.
We denote by $F(t)$ the **value** of a corporate entity at time t.
$F(t)$ consists of two parts:

$$F(t) = E(t) + B(t),$$

where
$E(t)$ is the corporate entity's equity
and $B(t)$ is its debt.

15.1. The Merton model

Remarks.

1. A corporate entity is an organization (e.g. enterprise, institution, firm, government agency, etc) that is recognized as having privileges and obligations, such as having the ability to enter into contracts, to sue, and to be sued.
2. In the above context, the *value* of a corporate entity is the same as the *total capital* of the corporate entity.
3. *Equity* or *shareholders' equity* is the part of the total capital of a business which belongs to the business (while debt doesn't).

15.1. The Merton model

Definition 15.5 (The Merton model)

The Merton model is the one that assumes that:

1. At time $t = 0$, the corporate entity's capital consists of equity $E(0)$ and debt $B(0)$ (that is $F(0) = E(0) + B(0)$).
   The equity $E(0)$ is owned by the shareholders
   and its debt $B(0)$ is the cost of the zero coupon bonds sold by the corporate entity.
2. The corporate entity promises to pay to bondholders the amount L at future time T.

15.1. The Merton model

If $F(T) \geq L$, that is at time T the total value of the corporate entity is greater than (or equal to) its debt L to the bondholders, then the bondholders receive L and the shareholders receive $F(T) - L$.

But if $F(T) < L$ then the corporate entity defaults, the bondholders receive $F(T)$ and the shareholders receive nothing.

So, the payoff to these two categories of investors will be:

$$\text{Shareholders:} \quad R_{sh}(T) = \max[F(T) - L, 0] = (F(T) - L)^{+} \tag{1}$$

$$\text{Bondholders:} \quad R_{bh}(T) = \min[F(T), L] = F(T) - (F(T) - L)^{+} \tag{2}$$

15.1. The Merton model

Exercise. Prove that for any real numbers x, y the following is true:
$\min[x,\, y] = y - (y - x)^{+} = x - (x - y)^{+}$.

Remark. The inequality $L > B(0)$ has to be satisfied because if it were otherwise then buying such a bond would have been a meaningless investment for the bondholder.

Exercise. Suppose that the interest rate compounded continuously is r. Prove that then the following stronger inequality holds: $B(0) \leq e^{-rT} L$.
Hint. Note that $R_{bh}(T) \leq L$ and use the fact that $B(0) = e^{-rT}\tilde{\mathbb{E}}(R_{bh}(T))$.

15.1. The Merton model

Merton made a simple but important observation:
The payoff function $R_{sh}(T) = \max[F(T) - L, 0] = (F(T) - L)^+$ is exactly the payoff function for a European call option $\mathrm{Call}(L, T)$ on the underlying capital $F(t)$ of the corporate entity.
This means that the shareholders of the corporate entity are treated as having a European call option $\mathrm{Call}(L, T)$. This implies the following statement.

Lemma 15.1

Suppose that the interest rate compounded continuously is r. Then

$$E(0) = e^{-rT}\tilde{\mathbb{E}}(F(T) - L)^+, \tag{3}$$

where $\tilde{\mathbb{E}}$ is the expectation with respect to the risk-neutral probability.

15.1. The Merton model

Proof.
By Theorem 5.2, the price C of a derivative maturing at time T and having a payoff function $R(T)$ is given by $C = e^{-rT}\tilde{\mathbb{E}}(R(T))$.
In our case, shares are treated as $\mathrm{Call}(L, T)$ European options with the payoff function (1). Hence their price is

$$C = e^{-rT}\tilde{\mathbb{E}}(F(T) - L)^{+}.$$

On the other hand, according to the definition of the model, the cost of the shares is $E(0)$. So, $C = E(0)$ and this implies (3).□

Remark. The derivation of equation (2) does not rely on any specific properties of the process $F(t)$. This means that this relation is **model-independent**.

$$\text{Bondholders:} \quad R_{bh}(T) = \min[F(T), L] = F(T) - (F(T) - L)^{+} \qquad (2)$$

15.1. The Merton model

Merton's observation and the Black-Scholes formula

All the statements made in the previous section are **independent** of any special properties of the process $F(t)$.
Here, we shall prove a theorem which makes use of Merton's observation in the context of the **Black-Scholes formula**.
This theorem states an equation which establishes the **dependence** between the price of the bond $B(0)$ and the equity value $E(0)$.

15.1. The Merton model

Theorem 15.1

Suppose that the total value $F(t)$ of the corporate entity evolves according to the geometric Brownian motion, $F(t) = F(0)\mathrm{e}^{\mu t+\sigma W(t)}$, and that the interest rate compounded continuously is r. Then

$$E(0) = (E(0) + B(0))\Phi(\omega) - Le^{-rT}\Phi(\omega - \sigma\sqrt{T})\,, \tag{4}$$

where

$$\omega = \frac{\log\frac{(E(0)+B(0))}{L} + rT}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}.$$

15.1. The Merton model

Proof.
Recall that within the framework of the Black-Scholes model the price C of a European call option $\mathrm{Call}(K, T)$ (that is with the strike price K and expiration time T) is given by

$$C = e^{-rT}\tilde{\mathbb{E}}(F(T) - K)^{+} = S\Phi(\omega) - Ke^{-rT}\Phi(\omega - \sigma\sqrt{T})\,, \tag{5}$$

where

$$\Phi(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \mathrm{e}^{-t^2/2}\,\mathrm{d}t \quad \text{and} \quad \omega = \frac{\log\frac{S}{K} + rT}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}.$$

In our case $S = F(0) = E(0) + B(0)$, $K = L$, and, as has been explained above, $C = E(0)$. Replacing S, K, and C in (5) by these values we obtain (4).$\square$

15.1. The Merton model

Corollary 15.1

If we know $E(0)$ (which may often be the case) then there is just one unknown variable in (4), namely $B(0)$. We can solve (4) numerically and thus compute $B(0)$.
Similarly, if $B(0)$ is known then $E(0)$ can be computed as solution to (4).
Finally, if $F(0)$ is known then we compute $E(0)$ using (4) and find $B(0) = F(0) - E(0)$.

Corollary 15.2

If we know $F(0)$ then we can find the probability of default:

$$\mathbb{P}(\text{default}) = \mathbb{P}(F(T) < L).$$

15.2. Two-state intensity-based model for credit ratings

A two-state model for credit rating is the simplest example of the so called reduced-form model.
An intensity-based model is a particular type of continuous time reduced-form model which is defined as follows. (Compare with what you've learnt in Survival Models.)

Definition 15.6

A two-state model assumes that:

- At each time t, a corporate entity can be in one of two states:
  - N =not previously defaulted;
  - D =defaulted.

  Denote by $X(t)$ the state of the corporate at time t, that is either $X(t) = N$ or $X(t) = D$.

15.2. Two-state intensity-based model for credit ratings

Definition 15.7 (cont.)

- There is a function $\lambda(t) \geq 0$ such that for $\Delta t > 0$ the following relations hold:

$$\mathbb{P}(X(t + \Delta t) = N \,\big|\, X(t) = N) = 1 - \lambda(t)\Delta t + o(\Delta t) \tag{6}$$

and

$$\mathbb{P}(X(t + \Delta t) = D \,\big|\, X(t) = N) = \lambda(t)\Delta t + o(\Delta t),$$

where $o(\Delta t)$ is the so called "o small of Δt": $\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0$.
$\lambda(t)$ is called the **transition intensity** from N to D.

15.2. Two-state intensity-based model for credit ratings

Definition 15.8 (cont.)

- Let τ be the time of default. If the corporate defaults then the bond payments are reduced by a deterministic factor:

$$\text{payment} = \begin{cases} 1 & \text{if } \tau > T \text{ (no default by time } T\text{)}, \\ \delta & \text{if } \tau \leq T \text{ (default takes place by time } T\text{)}, \end{cases}$$

  where $0 \leq \delta < 1$, T is the maturity time of the bond.
- The interest rate compounded continuously is r (and does not depend on t).

15.2. Two-state intensity-based model for credit ratings

Probability of default and the distribution of the time of default

The default of a bond takes place if $\tau \leq T$ and so the probability of default is

$$\mathbb{P}(\text{default}) = \mathbb{P}(\tau \leq T).$$

Recall that the cumulative distribution function of the time of default τ is defined by $F_\tau(t) = \mathbb{P}(\tau \leq t)$. We shall prove the following theorem.

15.2. Two-state intensity-based model for credit ratings

Theorem 15.2

For $t \geq 0$

$$F_\tau(t) = 1 - \mathrm{e}^{-\int_0^t \lambda(s)ds}. \tag{7}$$

Proof. Define $p(t) = \mathbb{P}(\tau > t)$. Obviously $F_\tau(t) = 1 - p(t)$ and hence, in order to prove Theorem 15.2 it suffices to prove following equivalent statement:

$$p(t) = \mathrm{e}^{-\int_0^t \lambda(s)ds} \tag{8}$$

In turn, (8) will be deduced from the following lemma.

Lemma 15.2

$$p'(t) = -\lambda(t)p(t) \tag{9}$$

15.2. Two-state intensity-based model for credit ratings

Proof (cont). It remains to establish (8). Rewrite (9) as

$$\frac{p'(t)}{p(t)} = -\lambda(t) \text{ or, equivalently } (\ln(p(t))' = -\lambda(t).$$

Integrating the last relation gives

$$\int_0^t \ln(p(s))' ds = -\int_0^t \lambda(s) ds \text{ and hence } \ln(p(t) - \ln(p(0) = -\int_0^t \lambda(s) ds.$$

It follows that $\frac{p(t)}{p(0)} = \exp(-\int_0^t \lambda(s)ds)$ and so

$$p(t) = p(0)\mathrm{e}^{-\int_0^t \lambda(s)ds}.$$

Note that $p(0) = 1$ because the time of default is always strictly positive (no corporate can start its existence by defaulting). This proves (8). The theorem is now proven.□

15.2. Two-state intensity-based model for credit ratings

Proof of Lemma 15.2.

Note that for $\Delta t > 0$

$$p(t+\Delta t) = \mathbb{P}(\tau > t+\Delta t) \overset{(*)}{=} \mathbb{P}(\tau > t+\Delta t \text{ and } \tau > t) \overset{(**)}{=} \mathbb{P}(\tau > t)\mathbb{P}(\tau > t+\Delta t \,|\, \tau > t).$$

We use here two facts which you know from the Introduction to Probability course:

(a) $(*)$ follows from $\mathbb{P}(A) = \mathbb{P}(A \cap B)$ if $A \subset B$

(b) $(**)$ follows from $\mathbb{P}(A \cap B) = \mathbb{P}(B)\mathbb{P}(A \,|\, B)$ for any two events A and B.

Here $A = \{\tau > t + \Delta t\}$, $B = \{\tau > t\}$).

It follows that

$$\begin{aligned} p(t+\Delta t) &= p(t)\mathbb{P}(\tau > t+\Delta t \,|\, \tau > t) \\ &= p(t)\mathbb{P}(X(t+\Delta t) = N \,|\, X(t) = N) = p(t)(1 - \lambda(t)\Delta t + o(\Delta t)), \end{aligned}$$

where we use the equality of events $\{\tau > t\} = \{X(t) = N\}$ and the definition of the model (Eq (6)).

15.2. Two-state intensity-based model for credit ratings

Proof of Lemma 15.2 (cont).
We thus have proved that

$$p(t+\Delta t) = p(t)(1-\lambda(t)\Delta t + o(\Delta t)) = p(t) - \lambda(t)p(t)\Delta t + o(\Delta t).$$

Rearranging this equality we obtain

$$\frac{p(t+\Delta t)-p(t)}{\Delta t} = -\lambda(t)p(t) + \frac{o(\Delta t)}{\Delta t}.$$

Taking the limit of both parts of the last equation as $\Delta t \to 0$ we obtain

$$p'(t) = \lim_{\Delta t\to 0}\frac{p(t+\Delta t)-p(t)}{\Delta t} = -\lambda(t)p(t) + \lim_{\Delta t\to 0}\frac{o(\Delta t)}{\Delta t}$$

or $p'(t) = -\lambda(t)p(t)$. Lemma is proved.□

15.2. Two-state intensity-based model for credit ratings

Corollary 15.3

The probability of default is given by

$$\mathbb{P}(default) = \mathbb{P}(\tau \leq T) = 1 - \mathrm{e}^{-\int_0^T \lambda(s)ds}. \tag{10}$$

15.2. Two-state intensity-based model for credit ratings

Remarks.

1. We have computed $F_\tau(t)$ for $t \geq 0$. It is obvious that $F_\tau(t) = 0$ if $t < 0$.
   Exercise. Even though it is obvious, explain this statement to yourself.
2. If $\lambda > 0$ does not depend on t then $F_\tau(t) = 1 - \mathrm{e}^{-\lambda t}$ and the probability density function of τ is

$$f_\tau(t) = F'(t) = \lambda \mathrm{e}^{-\lambda t} \text{ if } t \geq 0 \text{ (and } f_\tau(t) = 0 \text{ if } t < 0).$$

   We thus see that τ is an exponential random variable, $\tau \sim \mathrm{Exp}(\lambda)$, if $\lambda > 0$ does not depend on t.
3. The probability that default will eventually happen is 1 if and only if $\int_0^\infty \lambda(s) ds = \infty$.
   Exercise. Prove this statement.

15.2. Two-state intensity-based model for credit ratings

Bonds in the framework of the two-state model
Let $B(t, T)$ be the (risk neutral) price at time t, $0 \leq t \leq T$, of the bond with the payoff function $R(T)$ defined at the beginning of this section:

$$R(T) = \begin{cases} 1 & \text{if } \tau > T \text{ (no default by time } T), \\ \delta & \text{if } \tau \leq T \text{ (default takes place by time } T), \end{cases}$$

The question is: **how to compute the** $B(t, T)$**?**

15.2. Two-state intensity-based model for credit ratings

By the general rule (Theorem 5.2), $B(t,T) = \mathrm{e}^{-rT}\tilde{\mathbb{E}}(R(T))$.
Here, as usual, $\tilde{\mathbb{E}}$ is the expectation over the risk-neutral probability.
To proceed, we need the following statement which we shall use without proof.

Statement. *There exists the risk-neutral intensity $\tilde{\lambda}(t)$ which can be used to compute prices of derivatives related to the two-state model.*

Suppose that $\tilde{\lambda}(t)$ is known to us and that the risk-neutral probabilities can be computed in the same way as the real life probabilities discussed above: the only difference is that $\lambda(t)$ should be replaced by $\tilde{\lambda}(t)$.

15.2. Two-state intensity-based model for credit ratings

Example.

The real-life probability of default is given by (10).

$$\mathbb{P}(\text{default}) = \mathbb{P}(\tau \leq T) = 1 - \mathrm{e}^{-\int_0^T \lambda(s)ds}. (10)$$

Hence, the risk-neutral probability of this event is

$$\tilde{\mathbb{P}}(\tau \leq T) = 1 - \mathrm{e}^{-\int_0^T \tilde{\lambda}(s)ds}. \tag{11}$$

Similarly,

$$\tilde{\mathbb{P}}(\tau > T) = \mathrm{e}^{-\int_0^T \tilde{\lambda}(s)ds}. \tag{12}$$

It is now easy to compute $B(0, T)$ in terms of $\tilde{\lambda}(s)$. Namely, $R(T)$ is a random variable taking values 1 and δ. Hence

$$\tilde{\mathbb{E}}(R(T)) = 1 \times \tilde{\mathbb{P}}(R(T) = 1) + \delta \times \tilde{\mathbb{P}}(R(T) = \delta).$$

15.2. Two-state intensity-based model for credit ratings

Example (cont).
Since $\tilde{\mathbb{P}}(R(T) = 1) = \tilde{\mathbb{P}}(\tau > T)$ and $\tilde{\mathbb{P}}(R(T) = \delta) = \tilde{\mathbb{P}}(\tau \leq T)$ we obtain (using (12)):

$$\tilde{\mathbb{E}}(R(T)) = \tilde{\mathbb{P}}(\tau > T) + \delta\tilde{\mathbb{P}}(\tau \leq T) = \mathrm{e}^{-\int_0^T \tilde{\lambda}(s)ds} + \delta(1 - \mathrm{e}^{-\int_0^T \tilde{\lambda}(s)ds}) = (1-\delta)\mathrm{e}^{-\int_0^T \tilde{\lambda}(s)ds} + \delta.$$

Finally,

$$B(0, T) = \mathrm{e}^{-rT}\tilde{\mathbb{E}}(R(T)) = \mathrm{e}^{-rT}\left((1-\delta)\mathrm{e}^{-\int_0^T \tilde{\lambda}(s)ds} + \delta\right).$$

15.2. Two-state intensity-based model for credit ratings

How can we compute $B(t, T)$ when $0 < t \leq T$?
Exercise. Prove that

$$B(t,T) = \mathrm{e}^{-r(T-t)} \left((1-\delta)\mathrm{e}^{-\int_t^T \tilde{\lambda}(s)ds} + \delta \right).$$

Hint. You can view the whole process as starting at time t rather than 0 and take into account that in this case the duration of the process is $T - t$.

15.3. The Jarrow-Lando-Turnbull (JLT) model

The JLT model is an example of a more realistic **reduced-form** model which describes the behaviour of the ratings of bonds.

Ratings of bonds are provided by well-established rating agencies, such as **Standard & Poor's** (S&P) and **Moody's**. E.g., the Standard&Poor's ratings are

$$AAA, \; AA, \; A, \; BBB, \; BB, \; B, \; CCC, \; D,$$

where AAA is the best value of the rating, AA is the next one, ... , and D means default.

Example In 2021 one of the Barclays' bonds was rated BBB by the Standard&Poor's.
Remark. The above list of possible values of a rating is just an example which is sufficient for our purposes. In reality, S&P provide also more finely tuned values of a rating such as $AAA+$, $AAA-$, etc.

15.3. The Jarrow-Lando-Turnbull (JLT) model

Definition of the JLT model

Suppose that the rating of a bond can take n different values: 1, 2, ..., $n-1$, n, where 1 corresponds to the best rating, 2 correspond the next one, ..., n corresponds to D (default).

In the above example $n = 8$ with rating 1 corresponding to AAA, 2 corresponding to AA, ..., 7 corresponding to CCC, and 8 corresponding to D.

Denote by $X(t)$ the rating of the bond at time $t \geq 0$. So, $X(t)$ takes one of the values from the range $1, 2, ..., n$.

As time progresses, the rating may change, say from $X(s) = i$ to $X(t) = j$ (where $t \geq s$).

Let $p_{ij}(s,t)$ be the conditional probability of the event that the rating of the bond at time t will be j given that at time s, $s \leq t$, it is i:

$$p_{ij}(s,t) = \mathbb{P}(X(t) = j \,\big|\, X(s) = i), \quad \text{where } 1 \leq i, j \leq n; \quad s \leq t.$$

15.3. The Jarrow-Lando-Turnbull (JLT) model

Definition 15.9 (The Jarrow-Lando-Turnbull (JLT) model)

The JLT model assumes that for $\Delta t \geq 0$

$$p_{ij}(t, t+\Delta t) = \lambda_{ij}(t)\Delta t + o(\Delta t) \quad \text{if} \quad i \neq j,$$
$$p_{ii}(t, t+\Delta t) = 1 - \lambda_{ii}(t)\Delta t + o(\Delta t)$$

where $\lambda_{ij}(t) \geq 0$ are the **transition intensities** satisfying

$$\lambda_{ii}(t) = \sum_{1 \leq j \leq n, j \neq i} \lambda_{ij}(t). \tag{13}$$

The following very important fact is proven in the theory of Markov chains:
Statement If $\lambda_{ij}(t)$ are known then $p_{ij}(s, t)$ can be computed.

15.3. The Jarrow-Lando-Turnbull (JLT) model

Remarks.

1. The JLT model is used for solving problems similar to the ones discussed in the previous section (e.g., computing the probabilities of default and the related bond prices).
2. Those who are familiar with the theory of random processes may have noticed that the JLT model is a particular example of a continuous time Markov chain.
3. It is obvious that $\sum_{j=1}^{n} p_{ij}(s,t) = 1$ (but do explain this statement!). This equality implies that (13) is satisfied.
   Exercise. Prove this fact.
4. By the definition of default, $p_{n,n}(s,t) = 1$ (and hence $p_{n,j}(s,t) = 0$ for $j \neq n$). In terms of the theory of Markov chains, n is the so called **absorbing state**: if at some (random) moment τ the process X reaches n, $X(\tau) = n$, then it remains in this state for ever.
5. A more advanced theory of credit risk deals also with intensities $\lambda_{ij}(t)$ which are themselves random processes.