Actuarial Financial Engineering

Week 8

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January 20, 2023

Overview of this week

13. Stochastic Process

13.1. The 'usual' differential of a function

Suppose $F(x)$ is a function $F : \mathbb{R} \to \mathbb{R}$ and $F'(x)$ is continuous.

Definition 13.1

$dF(x) = F'(x)dx$ (here dx is "small").

Explanation. $dF(x)$ is the linear part of the increment $\Delta F(x) = F(x + \Delta x) - F(x)$. By the Taylor formula,

$$F(x + dx) = F(x) + F'(x)dx + \frac{1}{2}F''(\theta)dx^2, \tag{1}$$

where θ is (unknown) point in $(x, x + dx)$ if $dx > 0$ and $\theta \in (x + dx, x)$ if $dx < 0$.

13.1. The 'usual' differential of a function

Explanation (cont). The important fact is that the difference between $\Delta F(x) = F(x + dx) - F(x)$ and $dF(x) = f'(x)dx$ is much smaller than dx (when dx is a small number). More precisely,

$$\frac{\Delta F(x) - dF(x)}{dx} \to 0 \quad \text{as} \quad dx \to 0.$$

Indeed, it follows from (1) that
$\Delta F(x) - dF(x) = F(x + dx) - F(x) - F'(x)dx = \frac{1}{2}F''(\theta)dx^2$ and hence

$$\frac{\Delta F(x) - dF(x)}{dx} = \frac{1}{2}F''(\theta)dx \to 0 \quad \text{as} \quad dx \to 0.$$

13.1. The ‘usual’ differential of a function

Example.
$F(x) = \sqrt{x}$. Then $F(1) = 1$, $F'(x) = \frac{1}{2}x^{-\frac{1}{2}}$, $F'(1) = \frac{1}{2}$.

$$\Delta F(1) = F(1 + dx) - F(1) \simeq F'(1)dx.$$

Since $F(x + dx) - F(x) \simeq dF(x)$, we have

$$F(1 + 0.05) = F(1) + dF(1), \quad \text{(with } dx = 0.05).$$

That is

$$\sqrt{1 + 0.05} \simeq 1 + dF(1) = 1 + \frac{0.05}{2} = 1.025.$$

13.2. Ito's formula (Ito's formula for $F(W_t)$)

Question. What is $dF(W_t)$? Here $F : \mathbb{R} \to \mathbb{R}$ and W_t is the standard Wiener process.

Note that if $g(x)$ is a differentiable function, then

$$dF(g(x)) = F'(g(x))g'(x)dx \tag{2}$$

However

$$dF(W(t)) \neq F'(W(t))\frac{dW(t)}{dt}dt,$$

since the derivative $\frac{dW(t)}{dt}$ does not exists.

13.2. Ito's formula (Ito's formula for $F(W_t)$)

Next, (2) can be rewritten as

$$dF(g(x)) = F'(g(x))dg(x), \quad \text{since } dg(x) = g'(x)dx. \tag{3}$$

Can we state that

$$dF(W(t)) = F'(W(t))dW(t)?$$

The answer is NO! The correct answer is given by **Ito's lemma**.

13.2. Ito's formula (Ito's formula for $F(W_t)$)

Lemma 13.1 (Ito's lemma)

Let $F(x)$ be a function $F : \mathbb{R} \to \mathbb{R}$ which has two derivatives $F'(x), F''(x)$ and $F''(x)$ is continuous.
Then

$$dF(W_t) = F'(W_t)dW_t + \frac{1}{2}F''(W_t)dt. \tag{4}$$

Remark. By definition, $dW_t \equiv \Delta W_t \equiv W(t + dt) - W(t)$.

The main explanation of the Ito formula is due to the following theorem.

Theorem 13.1

Suppose that $F(x)$ has two continuous and bounded derivatives: $F'(x), F''(x)$.
Then

$$F(W(b)) - F(W(a)) = \int_a^b F'(W_s)dW_s + \frac{1}{2}\int_a^b F''(W_s)ds. \tag{5}$$

13.2. Ito's formula (Ito's formula for $F(W_t)$)

(Note: we shall not prove this theorem but you are supposed know this statement.) Let us now compare (5) with the following relation which you have discussed in the Calculus courses. Namely, you know of course that

$$F(b) - F(a) = \int_a^b F'(x)dx.$$

Moreover, if a function $g(x)$, $g : \mathbb{R} \mapsto \mathbb{R}$, has a continuous derivative $g'(x)$ then

$$F(g(b)) - F(g(a)) = \int_a^b F'(g(x))g'(x)dx = \int_a^b F'(g(x))dg(x) \text{ (since } dg(x) = g'(x)dx).$$

However, (5) tells us that

$$F(W(b)) - F(W(a)) \neq \int_a^b F'(W_t)dW_t.$$

(And this happens because $W'(t)$ does not exist!)

13.2. Ito's formula (Ito's formula for $F(W_t)$)

One useful corollary of the Ito formula:

Corollary 13.1

Equation (5) *can be rearranged as follows:*

$$\int_a^b F'(W_s)dW_s = F(W(b)) - F(W(a)) - \frac{1}{2}\int_a^b F''(W_s)ds. \quad (6)$$

Example 1.
$\int_a^b dW_s = W_b - W_a$. Here $F(x) = x$, $F(W_t) = W_t$, $F'(W_t) = 1$. So

$$\int_a^b F'(W_s)dW_s = \int_a^b dW_s = W_b - W_a.$$

This is a particular case of (6).

13.2. Ito's formula (Ito's formula for $F(W_t)$)

Example 2.
$F(x) = x^2$. We have $F'(x) = 2x$, $F''(x) = 2$ and so (6) now reads

$$\int_a^b 2W_s dW_s = W_b^2 - W_a^2 - \frac{1}{2}\int_a^b 2ds = W_b^2 - W_a^2 - (b-a).$$

In particular,

$$\int_0^t W_s dW_s = \frac{1}{2}W_t^2 - \frac{1}{2}t.$$

13.2. Ito's formula (Ito's formula for $F(W_t)$)

IMPORTANT CONCLUSION

We know that, by definition,

$$\int_0^t f(W_s)dW_s = \lim_{\max_i \Delta t_i \to 0} \sum_{i=0}^{n-1} f(W_i)\Delta W_i.$$

To *compute* this *stochastic integral* in terms of the *ordinary integral* one can do the following:

1. Find $F(x)$ such that $F'(x) = f(x)$.
2. Then $\int_0^t f(W_s)dW_s = F(W_t) - F(0) - \frac{1}{2}\int_0^t f'(W_s)ds$.

This is what we did in the examples considered above.

Exercise. Compute the following stochastic integrals:

(a) $\int_0^t W_s^3 dW_s$

(b) $\int_0^t e^{W_s} dW_s$.

13.2. Ito's formula (Ito's formula for $F(W_t)$)

One more explanation of Ito's formula.

The material of this subsection is not examinable. It is here for those who want to know more. By Taylor's formula,

$$F(x + dx) - F(x) = F'(x)dx + \frac{1}{2}F''(x)dx^2 + \frac{1}{3!}F^{(3)}(\theta)dx^3 \tag{7}$$

As usual, θ is not known but this does not matter since we suppose that $F^{(3)}(x) = F'''(x)$ is bounded: $\mid F^{(3)}(x) \mid <$ Constant. We can use (7) (taking into account that $W(t + dt) = W(t) + dW(t)$) to obtain

$$F(W_t + dW_t) - F(W_t) = F'(W_t)dW_t + \frac{1}{2}F''(W_t)dW_t^2 + \frac{1}{3!}F^{(3)}(\theta)(dW_t)^3. \tag{8}$$

Note that $\mathbb{E}(dW_t^2) = \mathbb{E}((W_{t+dt} - W_t)^2) = dt$ (by the definition of the Wiener process). Note also that $\mathbb{E}(\mid dW_t \mid^3) = c(dt)^{3/2}$, where c is a constant.

13.2. Ito's formula (Ito's formula for $F(W_t)$)

So Ito's lemma (see Lemma 13.1) does the following:
It tells us that we can replace dW_t^2 in (8) by dt (i.e. $dW_t^2 = dt$),
and we can drop $(dW_t)^3$ since the expectation of $| \, dW_t \, |^3$ is much smaller than dt.

$$dF(W_t) = F'(W_t)dW_t + \frac{1}{2}F''(W_t)dt. \qquad (4)$$

13.2. Ito's formula (Ito's formula for $F(t, W_t)$)

Let $F(t,x)$ be a function of t and x, $F : \mathbb{R}^2 \to \mathbb{R}$.

Lemma 13.2 (Ito's formula for $F(t, W_t)$)

$$dF(t, W_t) = \left(\frac{\partial F(t, W_t)}{\partial t} + \frac{1}{2} \frac{\partial^2 F(t, W_t)}{\partial W_t^2} \right) dt + \frac{\partial F(t, W_t)}{\partial W_t} dW_t \tag{9}$$

13.2. Ito's formula (Ito's formula for $F(t, W_t)$)

Remarks.

1. Here and throughout the rest of the module, we assume that all the derivatives we need **exist**, are **continuous** functions, and have all the properties we may want them to have.
2. Even though the notations we use in (9) should be easy to understand, here is an additional explanation of their meaning:

$$\frac{\partial F(t, W_t)}{\partial W_t} = \frac{\partial F(t,x)}{\partial x} \Big|_{x=W_t}, \quad \frac{\partial^2 F(t, W_t)}{\partial W_t^2} = \frac{\partial^2 F(t,x)}{\partial x^2} \Big|_{x=W_t}.$$

Example. $F(t,x) = t^2 + x^2$. We have $\frac{\partial F}{\partial t} = 2t$, $\frac{\partial F}{\partial x} = 2x$, $\frac{\partial^2 F}{\partial x^2} = 2$. So

$$dF(t, W_t) = (2t+1)dt + 2W_t dW_t.$$

13.2. Ito's formula (The chain rule)

Suppose that Y_t is a stochastic process and that

$$dY_t = a(t, Y_t)dt + \sigma(t, Y_t)dW_t, \tag{10}$$

where a and σ are "good" functions. Then

$$dF(t, Y_t) = \left(\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial Y_t^2} + a\frac{\partial F}{\partial Y_t} \right) dt + \sigma \frac{\partial F}{\partial Y_t} dW_t. \tag{11}$$

Here

$$\frac{\partial F}{\partial t} \equiv \frac{\partial F(t, Y_t)}{\partial t},$$
$$\frac{\partial F}{\partial Y_t} \equiv \frac{\partial F(t, Y_t)}{\partial Y_t},$$
$$\frac{\partial^2 F}{\partial Y_t^2} \equiv \frac{\partial^2 F(t, Y_t)}{\partial Y_t^2}.$$

13.2. Ito's formula (The chain rule)

Note that Ito's formula for $F(t, W_t)$ is a particular case of the Chain rule:

$$dF(t, W_t) = \left(\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial W_t^2}\right) dt + \frac{\partial F}{\partial W_t} dW_t.$$

$$dF(t, Y_t) = \left(\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial Y_t^2} + a\frac{\partial F}{\partial Y_t}\right) dt + \sigma \frac{\partial F}{\partial Y_t} dW_t. \tag{11}$$

$\sigma = 1, \alpha = 0$

13.2. Ito's formula (How to remember (11) and similar formulae?)

1. We know Taylor's formula up to order 2:

$$dF(t,x) = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}dx + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}dx^2 + \frac{\partial^2 F}{\partial x \partial t}dxdt + \frac{1}{2}\frac{\partial^2 F}{\partial t^2}dt^2. \qquad (12)$$

   Here $\frac{\partial F}{\partial t} \equiv \frac{\partial F(t,x)}{\partial t}$, $\frac{\partial F}{\partial x} \equiv \frac{\partial F(t,x)}{\partial x}$, ...
2. Use the following formal rules when you replace x by W_t or Y_t:
   (a) $dW_t^2 = dt$;
   (b) $dtdW_t = 0$, $dt^2 = 0$. (Drop any items 'smaller' than dt. dt is very small.)

13.2. Ito's formula (How to remember (11) and similar formulae?)

Example 1.
Replace x in (12) by W_t. Then

$$\begin{aligned} dF(t, W_t) &= \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial W_t} dW_t + \frac{1}{2} \frac{\partial^2 F}{\partial W_t^2} dt + 0 + 0 \\ &= \left(\frac{\partial F(t, W_t)}{\partial t} + \frac{1}{2} \frac{\partial^2 F(t, W_t)}{\partial W_t^2} \right) dt + \frac{\partial F(t, W_t)}{\partial W_t} dW_t \end{aligned}$$

which is Ito's lemma for $F(t, W_t)$.

13.2. Ito's formula (How to remember (11) and similar formulae?)

Example 2.

Replace x in (12) by Y_t. Note that, according to the second rule

$$(dY_t)^2 = a^2dt^2 + 2a\sigma dtdW_t + \sigma^2 dW_t^2 = \sigma^2 dt.$$

Here we use equation (10). So

$$\begin{aligned} dF(t, Y_t) &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial Y_t}dY_t + \frac{1}{2}\frac{\partial^2 F}{\partial Y_t^2}dY_t^2 + 0 + 0 \\ &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial Y_t}(adt + \sigma dW_t) + \frac{1}{2}\frac{\partial^2 F}{\partial Y_t^2}\sigma^2 dt \end{aligned} \tag{13}$$

Hence the chain rule:

$$dF(t, Y_t) = \left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial Y_t}a + \frac{1}{2}\frac{\partial^2 F}{\partial Y_t^2}\sigma^2\right)dt + \sigma\frac{\partial F}{\partial Y_t}dW_t.$$

13.2. Ito's formula (How to remember (11) and similar formulae?)

Remark The (13) above contains two zeros. This is because

$$dtdY_t = dt \cdot (adt + \sigma dW_t) = 0 \quad \text{and} \quad dt^2 = 0$$

So

$$\frac{\partial^2 F(t, Y_t)}{\partial t \partial Y_t} dtdY_t = 0 \ \text{ and } \ \frac{\partial^2 F(t, Y_t)}{\partial t^2} dt^2 = 0$$

13.3. Stochastic differential equations

Definition 13.2 (Stochastic differential equation (SDE))

A stochastic differential equation (SDE) is the equation of the form

$$dY_t = a(t, Y_t)dt + \sigma(t, Y_t)dW_t, \tag{14}$$

where $a(t, Y_t)$, $\sigma(t, Y_t)$ are given (random) functions
and $Y_t = Y(t)$ is an unknown random process.

Remark. We have seen (14) before: equation (10).

13.3. Stochastic differential equations

Definition 13.3

We say that $Y(t)$ is a solution to (14) with initial value $Y(0)$, if for $t \geq 0$

$$Y(t) = Y(0) + \int_0^t a(s, Y_s)ds + \int_0^t \sigma(s, Y_s)dW_s. \tag{15}$$

Terminological remarks.

$Y(t)$ solving (14) is said to be a **diffusion process**.

$a(t, Y_t)$ is called the **drift** and $\sigma(t, Y_t)$ is the **volatility** of the diffusion process.

Note that (15) is obtained from (14) by integrating both parts of (14).

If $\sigma \equiv 0$, then (14) becomes $dY_t = a(t, Y_t)dt$ and is equivalent to $Y_t' = a(t, Y_t)$– the ordinary differential equation (but still, $Y(t)$ is a random process if a is a random process).

13.3. Stochastic differential equations

Simple examples of SDEs.

Example 1. The following relation is the simplest example of a SDE

$$dY_t = dW_t, \quad Y(0) = 0.$$

Then $Y_t = Y(0) + \int_0^t dW_s = W_t - W_0 = W_t$. Thus Y_t in this case is the Wiener process.

Example 2.

$$dY_t = \mu dt + \sigma dW_t, \quad Y(0) = 1,$$

where μ and σ are constants. Then

$$Y(t) = Y(0) + \int_0^t \mu ds + \int_0^t \sigma dW_s$$

and we obtain

$$Y(t) = 1 + \mu t + \sigma W_t,$$

which is the Brownian motion starting from 1.

13.3. Stochastic differential equations

Exercise 1.
$dY_t = e^{-t}dt + 2tdW_t$. State the distribution of Y_t if $Y(0) = -1$.
Exercise 2.
$dY_t = e^{-t}dt + 2tdW_t$. Find $d(Y_t^2)$.

13.3. Stochastic differential equations (SDE for the price of a share)

You are strongly advised to understand and be able to reproduce every step in the proofs explained below.
Let $S(t)$ be a random process describing the price of a share. How does the difference between $S(t)$ and $S(t+dt)$ behave?
A simple model for $dS(t) = S(t+dt) - S(t)$ is

$$dS(t) = S(t) \cdot adt + S(t) \cdot \xi(dt), \tag{16}$$

where a is a parameter (usually $a > 0$) and $\xi(dt)$ is a random "noise". The term $S(t) \cdot adt$ pushes the price up, while $\xi(dt)$ may be ≥ 0 or < 0. We choose $\xi(dt) = \sigma dW_t$, where W_t is the standard Wiener process and σ is a constant (which may be negative).
Then we obtain the following stochastic differential equation (SDE) :

$$dS(t) = aS(t)dt + \sigma S(t)dW_t \tag{17}$$

Assuming that $S(0) = S_0$ is given, how do we solve this SDE?

13.3. Stochastic differential equations (SDE for the price of a share)

The first solution to (17).

Theorem 13.2

The solution to (17) *is given by*

$$S_t = S_0 e^{(a - \frac{\sigma^2}{2})t + \sigma W_t}.$$

Proof. Rewrite (17) as follows:

$$\frac{dS_t}{S_t} = a dt + \sigma dW_t \quad \text{with} \quad S(0) = S_0 \tag{18}$$

Note that the left hand side of (18) resembles the differential $d \ln S(t)$ (but in fact it is not equal to this differential as will be seen below).
So, let us compute $d \ln S(t)$ using the chain rule version of Ito's lemma.

13.3. Stochastic differential equations (SDE for the price of a share)

Proof (cont). Recall that the differential of a function $F(S_t)$ (which is is good enough, say has two continuous derivatives) can be computed as follows:

$$dF(S_t) = F'(S_t)dS_t + \frac{1}{2}F''(S_t)(dS_t)^2.$$

In our case $F(x) = \ln x$ and so $F'(x) = (\ln x)' = \frac{1}{x}$, $F''(x) = (\ln x)'' = -\frac{1}{x^2}$ and $(dS_t)^2 = \sigma^2 S_t^2 dt$ (according to (17)). Hence

$$d\ln(S_t) = \frac{1}{S_t}(aS_t dt + \sigma S_t dW_t) - \frac{1}{2}\frac{1}{S_t^2} \times \sigma^2 S_t^2 dt = (a - \frac{\sigma^2}{2})dt + \sigma dW_t.$$

Remark. We now see that indeed $d\ln(S_t) \neq adt + \sigma dW_t$.

13.3. Stochastic differential equations (SDE for the price of a share)

Proof (cont). Integrating both parts of the last display formula, we obtain

$$\int_0^t d\ln(S_u) = \int_0^t ((a-\frac{\sigma^2}{2})du + \sigma dW_u) = \int_0^t (a-\frac{\sigma^2}{2})du + \int_0^t \sigma dW_u$$

and hence

$$\ln(S_t) - \ln(S_0) = (a-\frac{\sigma^2}{2})t + \sigma W_t$$

or, equivalently,

$$\frac{S_t}{S_0} = e^{(a-\frac{\sigma^2}{2})t+\sigma W_t} \quad \text{and} \quad S_t = S_0 e^{(a-\frac{\sigma^2}{2})t+\sigma W_t}.$$

□

13.3. Stochastic differential equations (SDE for the price of a share)

The second solution to (17) **(not examinable).**
This solution is slightly more difficult than the first one; it can be viewed as a demonstration of the usefulness of the Ito formulae.
Plan: the main steps of the second solution.

1. Suppose that $S(t)$ can be found in the form $S(t) = f(t, W_t)$, where $f(t, x)$ is a function of two variables, t and x.
2. Use Ito's lemma and substitute $S(t)$ in (17) by $f(t, W_t)$ and $dS(t)$ by $df(t, W_t)$.
3. Then see whether you can find $f(t, x)$.

13.3. Stochastic differential equations (SDE for the price of a share)

Theorem 13.3

$f(t,x) = S_0 e^{\mu t + \sigma x}$, where $\mu = a - \frac{\sigma^2}{2}$ and $S_0 = S(0)$.

Proof. Step 1. By Ito's lemma,

$$df(t,W_t) = \left(\frac{\partial f(t,W_t)}{\partial t} + \frac{1}{2} \frac{\partial^2 f(t,W_t)}{\partial W_t^2} \right) dt + \frac{\partial f(t,W_t)}{\partial W_t} dW_t \tag{19}$$

Substituting the left side of (17) by (19) we get

$$\left(\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial W_t^2} \right) dt + \frac{\partial f}{\partial W_t} dW_t = afdt + \sigma f dW_t \,, \tag{20}$$

where we write f for $f(t,W_t)$.

13.3. Stochastic differential equations (SDE for the price of a share)

Proof (cont). Equating the coefficients in front of dW_t in both sides of (20), we get

$$\frac{\partial f(t, W_t)}{\partial W_t} = \sigma f(t, W_t) \tag{21}$$

Rewrite (21) as

$$f'_x(t, x) = \sigma f(t, x) \tag{22}$$

We use here the notation $f'_x = \frac{\partial f}{\partial x}$. Fix t, then (22) is the simplest linear equation (known to you from the course Differential Equations). It has the general solution of the form

$$f(t, x) = c(t)e^{\sigma x}. \tag{23}$$

Remark: you can check this by substituting this expression into (22). Do it!
Note that $c(t)$ in (23) is an unknown function of t. It remains to find it.

13.3. Stochastic differential equations (SDE for the price of a share)

Proof (cont). Step 2. To find $c(t)$, we shall use another relation which follows from (20). Namely, we equate the coefficients in front of dt on both sides of (20) and get

$$f'_t(t,x) + \frac{1}{2} f''_{xx}(t,x) = af(t,x). \tag{24}$$

Next, it follows from (23) that

$$f'_t(t,x) = c'(t)e^{\sigma x} \tag{25}$$

$$f''_{xx}(t,x) = \sigma^2 c(t)e^{\sigma x} \tag{26}$$

Substituting (25) and (26) into (24) we get

$$c'(t)e^{\sigma x} + \frac{1}{2}\sigma^2 c(t)e^{\sigma x} = ac(t)e^{\sigma x}$$

and so

$$c'(t) = (a - \frac{\sigma^2}{2})c(t) \tag{27}$$

13.3. Stochastic differential equations (SDE for the price of a share)

Proof (cont). Hence $c(t) = c_0 e^{(a-\frac{\sigma^2}{2})t}$, where $c_0 = c(0)$. Finally,

$$f(t,x) = c_0 e^{\mu t + \sigma x}, \quad \text{where } \mu = a - \frac{\sigma^2}{2}.$$

□

We have thus proved that $S(t)$ can be found in the form $S(t) = f(t, W_t)$, namely:

$$S(t) = f(t, W_t) = c_0 e^{\mu t + \sigma W_t}.$$

Since $S(0) = c_0$, we get $c_0 = S_0$ and finally

$$S(t) = S_0 e^{\mu t + \sigma W_t}.$$

13.3. Stochastic differential equations (SDE for the price of a share)

Remarks.

1. We use the following fact: if
$$y'(x) = \alpha y(x) \tag{28}$$
then
$$y(x) = ce^{\alpha x}, \quad \text{where } c \text{ is a constant.} \tag{29}$$
2. If c in (29) depends on, say, t (as in (23)) then this means that we are, for some reason, considering a "family of solutions" with t being the parameter of the family.
3. (28) and (29) were used to solve (22) and (27). They will be used also in the next example.

13.3. Stochastic differential equations (The Ornstein-Uhlenbeck process)

Definition 13.4 (The Ornstein-Uhlenbeck process (OUP))

We say that $r(t)$ is the OUP if

$$dr = -a(r - \mu)dt + \sigma dW_t \tag{30}$$

where a, μ, σ are the parameters of the model.

We shall solve this equation for arbitrary (constant) parameters but in our applications the parameters a, b will be positive: $a > 0$, $\mu > 0$. Usually also $\sigma > 0$, but this is less important.

13.3. Stochastic differential equations (The Ornstein-Uhlenbeck process)

Before solving (30), let us consider the case when $\sigma = 0$. We then have $dr = -a(r-\mu)dt$, and since $dr = r'dt$ we obtain the following ordinary differential equation:

$$r' = -a(r-\mu). \tag{31}$$

Then $(r-\mu)' = -a(r-\mu)$, (as $(r-\mu)' = r' - \mu' = r'$) and hence

$$r - \mu = ce^{-at}, \quad \text{or} \quad r(t) = \mu + ce^{-at}.$$

It is useful to note that if $a > 0$ then $e^{-at} \to 0$ as $t \to \infty$ and hence $r(t) \to \mu$. Note also that $r(t) = \mu$ is a solution to (31). (See the sketch of the graph of $r(t)$ in the hand-written version of the Week 9 Notes.)

If $a > 0$ then the solution $r(t) = \mu$ is the so called stable solution.

13.3. Stochastic differential equations (The Ornstein-Uhlenbeck process)

Theorem 13.4

Suppose that $r(t)$ is a random process which satisfies the equation

$$dr = -a(r - \mu)dt + \sigma dW_t.$$

Then

$$r(t) = b + (r(0) - \mu)e^{-at} + \sigma e^{-at} \int_0^t e^{as} dW_s.$$

13.3. Stochastic differential equations (The Ornstein-Uhlenbeck process)

Proof. We shall be looking for a function $u(t)$ such that

$$r(t) - \mu = u(t)e^{-at} \tag{32}$$

Then $u(t) = e^{at}(r(t) - \mu)$. By Ito's lemma, we compute

$$\begin{aligned} du(t) &= ae^{at}(r - \mu)dt + e^{at}dr \\ &= ae^{at}(r - \mu)dt + e^{at}(-a(r - \mu)dt + \sigma dW_t) \\ &= \sigma e^{at}dW_t \end{aligned}$$

Hence $\int_0^t du(s) = \sigma \int_0^t e^{as}dW_s$, or equivalently,

$$u(t) - u(0) = \sigma \int_0^t e^{as}dW_s \tag{33}$$

13.3. Stochastic differential equations (The Ornstein-Uhlenbeck process)

Proof (cont). It follows from (32) that $r(0) - \mu = u(0)$. So (33) can be rewritten as

$$u(t) = u(0) + \sigma \int_0^t e^{as} dW_s = r(0) - \mu + \sigma \int_0^t e^{as} dW_s$$

and we obtain (again due to (32), replace the $u(t)$) that

$$\begin{aligned} r(t) &= \mu + e^{-at}\left(r(0) - \mu + \sigma \int_0^t e^{as} dW_s \right) \\ &= r(0)e^{-at} + \mu(1 - e^{-at}) + \sigma e^{-at} \int_0^t e^{as} dW_s \\ &= (r(0) - \mu)e^{-at} + \mu + \sigma e^{-at} \int_0^t e^{as} dW_s \end{aligned}$$

□

13.3. Stochastic differential equations (The Ornstein-Uhlenbeck process)

Some comments.

1. The most important step in the proof of this theorem is the “guess” (32). There is a good reason for this guess but we shall not discuss it here. However, you are required to know and be able to reproduce the above.

13.3. Stochastic differential equations (The Ornstein-Uhlenbeck process)

2. To compute $du(t)$, we use the chain rule. In fact we derive it. Namely, if $dr = -a(r-\mu)dt + \sigma dW_t$ and $u = f(t,r)$, then

$$du = f'_t dt + f'_r dr + \frac{1}{2} f''_{rr} (dr)^2.$$

In our case $u(t) = f(t,r) = e^{at}(r-\mu)$ and therefore

$$\begin{aligned} f'_t &= \frac{\partial}{\partial t}(e^{at}(r-\mu)) = ae^{at}(r-\mu), \\ f'_r &= \frac{\partial}{\partial r}(e^{at}(r-\mu)) = e^{at}, \\ f''_{rr} &= 0. \end{aligned}$$

This explains the second step.

Remark. We use the notation $f'_t = \frac{\partial f}{\partial t}$, $f'_r = \frac{\partial f}{\partial r}$, and $f''_{rr} = \frac{\partial^2 f}{\partial r^2}$.