MTH6112 Actuarial Financial Engineering
Coursework Week 8

1. (a) State the definition of the 'usual' differential of a function $F(x)$.
   Solution $dF(x) = F'(x)dx$.

   (b) Use the relation $F(x+dx) - F(x) \approx dF(x)$ it to compute $e^{0.1}$ and $\ln 0.9$ (without a calculator).

   Then compute $e^{0.1}$ and $\ln 0.9$ using a calculator and compare the results.
   Solution We use the relation $F(x + dx) \approx F(x) + dF(x) = F(x) + F'(x)dx$.

   If $F(x) = e^x$ then it is natural to choose $x = 0$ and $dx = 0.1$. Then $F(0) = 1$ and $F'(0) = 1$. So $e^{0.1} \approx 1 + 0.1 = 1.1$

   A calculator in this case gives $e^{0.1} \approx 1.1052$. So the error of our approximation is ≤ 0.0052

   If $F(x) = \ln x$ then it is natural to choose $x = 1$ and $dx = -0.1$. Then $F(1) = 0$ and $F'(1) = 1$. So $\ln 0.9 \approx -0.1$.

   Using a calculator in this case gives $\ln 0.9 \approx -0.1054$. So the error of our approximation is ≤ 0.0054

2. Reed Section 13.2 in the Slides of Week 8. Pay attention to the examples.

   (a) State Ito's lemma for $F(W_t)$.
   Solution $dF(W_t) = F'(W_t)dW_t + \frac{1}{2}F''(W_t)dt$

   (b) Compute $dF(W_t)$ for $F(x) = x^3$ and for $F(x) = e^x$.
   Solution So, applying Ito's formula gives

$$d(W_t^3) = 3W_t^2 dW_t + 3W_t dt, \quad d(e^{W_t}) = e^{W_t}dW_t + \frac{1}{2}e^{W_t}dt.$$

   (c) Use the results obtained in (b) to compute the following integrals

$$\int_0^t (W_s)^2 dW_s \quad \text{and} \quad \int_0^t e^{W_s}dW_s.$$

   Solution 1 We obtain from (b):

$$W_t^2 dW_t = \frac{1}{3}d(W_t^3) - W_t dt, \quad e^{W_t}dW_t = d(e^{W_t}) - \frac{1}{2}e^{W_t}dt.$$

Integrating both parts of these equalities, we obtain

$$\int_0^t (W_s)^2 dW_s = \frac{1}{3}W_t^3 - \int_0^t W_t dt, \quad \int_0^t e^{W_s} dW_s = e^{W_t} - 1 - \frac{1}{2}\int_0^t e^{W_t} dt.$$

Solution 2 We know the following corollary of Ito's lemma (see the slides)

$$\int_a^b F'(W(t))dW(t) = F(W(b)) - F(W(a)) - \frac{1}{2}\int_a^b F''(W(s))ds.$$

If $F'(x) = x^2$ then $F(x) = \frac{1}{3}x^3 + C$ where C is an arbitrary constant and $F''(x) = 2x$. Hence

$$\int_0^t (W_s)^2 dW_s = \frac{1}{3}W_t^3 - \int_0^t W_s ds.$$

Similarly, if $F'(x) = e^x$ then $F(x) = e^x + C$, $F''(x) = e^x$. So

$$\int_0^t e^{W_s} dW_s = e^{W_t} - 1 - \frac{1}{2}\int_0^t e^{W_s} ds.$$

3. a) Consider a diffusion process $V(t)$, $t \geq 0$, satisfying the following stochastic differential equation:

$$dV_t = (a + b\cos t)V_t dt + \sigma V_t dW_t. \tag{1}$$

Solve this equation using the method explained in lectures.
Solution Dividing both parts of the equation by V_t we get:

$$\frac{dV_t}{V_t} = (a + b\cos t)dt + \sigma dW_t.$$

The ratio $\frac{dV_t}{V_t}$ resembles the differential of the function $F(V) = \ln(V)$. Since V_t is a diffusion process, Ito's lemma tells us how to compute the differential of this function (which will not be equal to $\frac{dV_t}{V_t}$)). Namely,

$$d\ln(V_t) = F'(V_t)dV_t + \frac{1}{2}F''(V_t)(dV_t)^2 = \frac{1}{V_t}dV_t - \frac{1}{2V_t^2}(dV_t)^2,$$

where we use the fact that $(\ln x)' = \frac{1}{x}$, $(\ln x)'' = -\frac{1}{x^2}$. From (1) and because $(dV_t)^2 = \sigma^2 V_t^2 dt$ (explain this!), we get:

$$d\ln(V_t) = \frac{1}{V_t}\times[(a+b\cos t)V_t dt+\sigma V_t dW_t]-\frac{1}{2V_t^2}\times\sigma^2 V_t^2 dt = (a-\frac{\sigma^2}{2}+b\cos t)dt+\sigma dW_t.$$

Integration the relation $d\ln(V_t) = (a - \frac{\sigma^2}{2} + b\cos t)dt + \sigma dW_t$ over the interval $[0,\, t]$, we obtain:

$$\int_0^t d\ln(V_s) = \int_0^t (a - \frac{\sigma^2}{2} + b\cos s)ds + \int_0^t \sigma dW_s$$

and hence $\ln(V_t) - \ln(V_0) = (a - \frac{\sigma^2}{2})t + b\sin t + \sigma W_t$. Finally,

$$V(t) = V_0 e^{(a - \frac{\sigma^2}{2})t + b\sin t + \sigma W(t)}.$$

b) Write down $V(t)$ with $V_0 = 2.8$, $a = 0.5$, $b = 0.1$, and $\sigma = 0.2$.

Solution We first compute $a - \frac{\sigma^2}{2} = 0.23$. Hence

$$V(t) = 2.8e^{0.23t + 0.1\sin t + 0.2W(t)}.$$

Remark Here is one more way to solve (1). Namely, equation (1) for V has the same form as Equation (17) (Slide 27, Week 8) for S. According to Theorem 13.3, If we find $\mu(t)$ such that

$$\mu'(t) + \frac{1}{2}\sigma^2 = a + b\cos t \tag{2}$$

then we can state that $V(t)$ coincides with $S(t)$ (with properly chosen s). Obviously, (2) implies that we can choose $\mu(t) = (a - \frac{\sigma^2}{2})t + b\sin t$ and then according to Theorem 13.2 and 13.3,

$$V(t) = se^{(a - \frac{\sigma^2}{2})t + b\sin t + \sigma W(t)}.$$

Since $V(0) = s = V_0$, we finally obtain:

$$V(t) = V_0 e^{(a - \frac{\sigma^2}{2})t + b\sin t + \sigma W(t)}.$$

This solution is shorter than the one obtained by the method suggested in question (a). However, it is more artificial: one has to know the form of the solution.