# MTH6112 Actuarial Financial Engineering
# Coursework Week 6

1. Consider a share with price $S(t)$, $0 \le t \le T$. Suppose that a proportional dividend on this share is paid continuously at rate $q$ and is reinvested into the share. The interest rate compounded continuously is $r$. Let $C$ be the price of the European call option $\mathrm{Call}(K, T)$ on this share and $P$ be the be the price of the European put option $\mathrm{Put}(K, T)$ on the same share.

   Prove that then the following Call-Put parity relation holds:

   $$C - P = e^{-qT} S(0) - e^{-rT} K.$$

   **Solution** We know the following fact: the prices $C$ and $P$ are given by

   $$C = e^{-rT} \tilde{\mathbb{E}}(S(T) - K)^+ \quad \text{and} \quad P = e^{-rT} \tilde{\mathbb{E}}(K - S(T))^+,$$

   where $\tilde{\mathbb{E}}$ is the expectation over the risk-neutral probability (defined on the space of all possible functions $S(t)$, $0 \le t \le T$). Hence

   $$C - P = e^{-rT} \tilde{\mathbb{E}}(S(T) - K)^+ - e^{-rT} \tilde{\mathbb{E}}(K - S(T))^+ = e^{-r(T-t)} \tilde{\mathbb{E}} \left[ (S(T) - K)^+ - (K - S(T))^+ \right]$$

   Since $x^+ - (-x)^+ = x$ we see that

   $$C - P = e^{-rT} \tilde{\mathbb{E}}(S(T) - K) = e^{-rT} \left( \tilde{\mathbb{E}}(S(T)) - K \right) = e^{-rT} \tilde{\mathbb{E}}(S(T)) - e^{-rT} K.$$

   Recall that $\tilde{\mathbb{E}}(S(T)) = e^{(r-q)T} S(0)$ and therefore

   $$C - P = e^{-qT} S(0) - e^{-rT} K.$$

2. Recall the following definition of the index and of its value $I(t)$.

   **Definition** For $n$ shares with prices $S_1(t)$, $S_2(t)$,..., $S_n(t)$ the index $I(t)$ is defined by

   $$I(t) = \omega_1 S_1(t) + \omega_2 S_2(t) + \cdots + \omega_n S_n(t),$$

   where $\omega_1$, $\omega_2$,..., $\omega_n$ are positive numbers such that $\sum_{j=1}^{n} \omega_j = 1$. The numbers $w_j$ are called *weights*.

(a) Suppose that, unlike in the theorem proved in the notes, the strike price $K_j$ for the $j^{\text{th}}$ option does depend on $j$. Moreover, suppose also that the weights $\omega_j$ do not necessarily satisfy the relation $\sum_{j=1}^{n} \omega_j = 1$.

Prove that if $C_j(K_j, t)$ is the price of the call option on the share $S_j(t)$, $j = 1, ..., n$, and $K = \sum_{j=1}^{n} \omega_j K_j$ then

$$C_I(K, t) \leq \sum_{j=1}^{n} \omega_j C_j(K_j, t).$$

(The notations we use clearly indicate that the expiration time of all options is $t$.)

**Solution** Below, we solve this problem for the case of a put option (case (b)). The solution for (a) is essentially the same as for (b). The only difference is that you have to replace $P$ by $C$, $K - I(t)$ by $I(t) - K$ and $K_j - S_j(t)$ by $S_j(t) - K_j$.

(b) State and prove a similar relation for put options.

**Solution** Let $P_I(K, t)$ be the price of the Put$(K, t)$ on the index. Then

$$P_I(K, t) \leq \sum_{j=1}^{n} \omega_j P_j(K_j, t).$$

**Proof** By the general theorem (you are supposed to quote Theorem 5.2 from Slides of Week 3),

$$P_I(K, t) = e^{-rt}\tilde{\mathbb{E}}(K - I(t))^+$$

Since

$$K - I(t) = \sum_{j=1}^{n} \omega_j K_j - \sum_{j=1}^{n} \omega_j S_j(t) = \sum_{j=1}^{n} \omega_j (K_j - S_j(t))$$

we get that

$$(K - I(t))^+ = \left( \sum_{j=1}^{n} \omega_j (K_j - S_j(t)) \right)^+ \leq \sum_{j=1}^{n} \omega_j (K_j - S_j(t))^+.$$

So

$$P_I(K, t) = e^{-rt}\tilde{\mathbb{E}}(K - I(t))^+ \leq e^{-rt} \sum_{j=1}^{n} \omega_j \tilde{\mathbb{E}}(K_j - S_j(t))^+ = \sum_{j=1}^{n} \omega_j P_j(K_j, t). \quad \square$$

3. (a) State the definition of the implied volatility.
**Solution** Implied volatility is the solution $\sigma$ of the equation

$$C(K, T, S, \sigma, r) = c,$$

where $K$, $S$, $T$, $r$ are known parameters and $c$ is the market price of the option (real life price of the option).

(b) Write down the expression for $\frac{\partial C}{\partial \sigma}$. Can you state that this derivative is non-negative?
**Solution** See Slides of Week 3&4:

$$\frac{\partial C}{\partial \sigma} = S\sqrt{T}\Phi'(\omega) = \frac{1}{\sqrt{2\pi}} S\sqrt{T} \mathrm{e}^{\frac{\omega^2}{2}}.$$

It is clear from this expression that $\frac{\partial C}{\partial \sigma} > 0$.

(c) Prove that implied volatility is uniquely defined (if it exists).
**Solution** Since $C(K, T, S, \sigma, r)$ is a monotone function of $\sigma$, the above equation has only one solution (if at all).

4. As usual, we denote by $W_t \equiv W(t)$ the values at time $t \geq 0$ of the (standard) Wiener process. You are reminded that by definition

$$\int_a^b f(s) dW_s = \lim_{\delta \to 0} \sum_{i=0}^{n-1} f(t_i) \Delta W_i,$$

where

$$a = t_0 < t_1 < ... < t_{n-1} < t_n = b, \;\; \delta = \max_{0 \leq i \leq n-1} t_{i+1} - t_i, \;\; \text{and } \Delta W_i = W(t_{i+1}) - W(t_i).$$

(a) Compute the integral $\int_0^3 f(s) dW_s$ in terms of the values of $W(t)$ for a function defined by $f(s) = \begin{cases} -1 & \text{if } 0 \leq s < 1, \\ 1 & \text{if } 1 \leq s < 2, \\ -1 & \text{if } 2 \leq s \leq 3. \end{cases}$

**Solution** We saw a similar example in lectures. Here is what we have done: if $f(s) = C$ when $s \in [a, b]$ then

$$\int_a^b C dW_s = C \lim_{\delta \to 0} \sum_{i=0}^{n-1} \Delta W_i = C \lim_{\delta \to 0} \sum_{i=0}^{n-1} (W(t_{i+1}) - W(t_i))$$
$$= C \lim_{\delta \to 0} (W(t_n) - W(t_0)) = C(W(b) - W(a)).$$

(Do understand the equality $\sum_{i=0}^{n-1}(W(t_{i+1})-W(t_i)) = (W(t_n)-W(t_0))$!)
Hence in our case

$$\begin{aligned}\int_0^3 f(s)dW_s &= \int_0^1 (-1)\,dW_s + \int_1^2 1\,dW_s + \int_2^3 (-1)\,dW_s \\ &= -(W(1)-W(0)) + (W(2)-W(1)) - (W(3)-W(2)) \\ &= -W(3) + 2W(2) - 2W(1).\end{aligned}$$

(b) What is the distribution of the integral from part (a)?

**Solution 1.** By the definition of the Wiener process, the random variables $-(W(1)-W(0))$, $(W(2)-W(1))$, $-(W(3)-W(2))$ are independent and have the standard normal distribution $\mathcal{N}(0, 1)$. Hence $\int_0^3 f(s)dW_s \sim \mathcal{N}(0, 3)$.

**Solution 2.** A theorem discussed in lectures states that

$$\int_a^b f(s)dW_s \sim \mathcal{N}\left(0, \int_a^b f(s)^2 ds\right).$$

Since in our case $\int_a^b f(s)^2 ds = \int_0^3 ds = 3$, we get $\int_0^3 f(s)dW_s \sim \mathcal{N}(0, 3)$.

(c) Suppose that $f(s) = \begin{cases} 2 & \text{if } 0 \le s < 1, \\ -2 & \text{if } s \ge 1. \end{cases}$

Compute $Y(t) = \int_0^t f(s)dW_s$ for all $t \ge 0$ (in terms of the values of $W$).

**Solution** If $0 \le t < 1$ then $Y(t) = \int_0^t 2dW_s = 2W(t)$. If $t \ge 1$ then

$$\begin{aligned}Y(t) &= \int_0^t 2dW_s = \int_0^1 2dW_s + \int_1^t (-2)\,dW_s \\ &= 2W(1) - 2(W(t) - W(1)) = -2W(t) + 4W(1)\end{aligned}$$

Exercise: Prove that $Y(t) = 2\tilde{W}(t)$, where $\tilde{W}(t)$ is a standard Wiener process.

5. Read the Slides of this week.

a) Find the distribution of the random variables $\int_0^t s^2 dW_s$ and $\int_0^t e^{-s} dW_s$.

**Solution** By Theorem 12.3,

$$\mathrm{Var}\left(\int_0^t s^2 dW_s\right) = \int_0^t s^4 ds = \frac{1}{5}t^5, \ \mathrm{Var}\left(\int_0^t e^{-s} dW_s\right) = \int_0^t e^{-2s} ds = \frac{1}{2}(1-e^{-2t}).$$

Hence, by Theorem 12.3 again, $\int_0^t s^2 dW_s \sim \mathcal{N}(0, \frac{1}{5}t^5)$ and $\int_0^t e^{-s} dW_s \sim \mathcal{N}(0, \frac{1}{2}(1-e^{-2t}))$.

b) Compute the variance of the random variables $\int_0^t W_s^2 dW_s$ and $\int_0^t e^{-W(s)} dW_s$.

**Solution** We use Theorem 12.5:

**Theorem** $\mathrm{Var}\left(\int_a^b f(W_s)dW_s\right) = \int_a^b \mathbb{E}[f(W_s)^2]ds$.

We thus have

$$\mathrm{Var}\left(\int_0^t W_s^2 dW_s\right) = \int_0^t \mathbb{E}[W_s^4]ds, \quad \mathrm{Var}\left(\int_0^t e^{-W(s)} dW_s\right) = \int_0^t \mathbb{E}[e^{-2W(s)}]ds$$

We know that $\mathbb{E}[W_s^4] = 3s^2$ (see Lemma 1.2, Week 1), $\mathbb{E}[e^{-2W(s)}] = e^{2s}$ (see the proof of Theorem 1.1, Week 1), and so

$$\mathrm{Var}\left(\int_0^t W_s^2 dW_s\right) = \int_0^t 3s^2 ds = t^3, \quad \mathrm{Var}\left(\int_0^t e^{-W(s)} dW_s\right) = \int_0^t e^{2s} ds = \frac{1}{2}(e^{2t} - 1).$$

6. Consider a random process $Y(t)$, $t \geq 0$, defined by $Y(t) = \int_0^t f(s)dW_s$.

(a) This process has independent increments. Prove the following particular case of this statement: if $0 < \tau_1 < \tau_2$ then the random variables $Y(\tau_1)$ and $Y(\tau_2) - Y(\tau_1)$ are independent.

**Hint.** This property is a corollary of the definition of the integral. You have to use the independence of the increments $\Delta W_i$ of the Wiener process.

**Solution** By the definitions of $Y(t)$ and of the integral, we have

$$Y(\tau_1) = \int_0^{\tau_1} f(s)dW_s = \lim_{\delta \to 0} \sum_{i=0}^{n-1} f(t_i)\Delta W_i,$$

where $W_i = W(t_{i+1}) - W(t_i)$ with $0 < t_i < t_{i+1} \leq \tau_1$. Next,

$$Y(\tau_2) - Y(\tau_1) = \int_{\tau_1}^{\tau_2} f(s)dW_s = \lim_{\delta \to 0} \sum_{j=0}^{n'-1} f(t'_j)\Delta W'_j.$$

where $W'_j = W(t'_{j+1}) - W(t'_j)$ with $\tau_1 \leq t'_j < t'_{j+1} \leq \tau_2$.
Since all $t_i \in [0, \tau_1]$ and all $t'_j \in [\tau_1, \tau_2]$, we see that the corresponding $W_i$ and $W'_j$ are independent random variables. Hence also the sums $\sum_{i=0}^{n-1}(\cdot)$ and $\sum_{j=0}^{n'-1}(\cdot)$ are independent random variables and so are their limits.

(b) Using the property stated in $(a)$ prove that if $t_1 < t_2$ then

$$\text{Cov}(Y(t_1), Y(t_2)) = \text{Var}(Y(t_1)).$$

**Solution** By definition

$$\begin{aligned}\text{Cov}(Y(t_1), Y(t_2)) &= \mathbb{E}(Y(t_1)Y(t_2)) - \mathbb{E}(Y(t_1))\mathbb{E}(Y(t_2)) \\ &= \mathbb{E}(Y(t_1)Y(t_2))\end{aligned}$$

where we use the fact that $\mathbb{E}(Y(t)) = 0$ (Theorem 12.3). Next,

$$\begin{aligned}\mathbb{E}(Y(t_1)Y(t_2)) &= \mathbb{E}(Y(t_1)(Y(t_2) - Y(t_1) + Y(t_1))) \\ &= \mathbb{E}(Y(t_1)(Y(t_2) - Y(t_1))) + \mathbb{E}(Y(t_1)^2) \\ &= \mathbb{E}(Y(t_1)^2) = \text{Var}(Y(t_1)).\end{aligned}$$

Note that we use the fact that $Y(t_2) - Y(t_1)$ and $Y(t_1)$ are independent!
Remark. This proof is essentially the same as the proof of a similar property of the Wiener process.

(c) Compute $\text{Cov}(Y(t_1), Y(t_2))$ in the case when $Y(t) = \int_0^t e^{t-s} dW_s$.

**Solution** Note that $Y(t) = e^t \int_0^t e^{-s} dW_s$. Since $\text{Cov}(aX, bZ) = ab\text{Cov}(X, Z)$ for any random variables $X$, $Y$ and any constants $a$, $b$, we obtain from this property and 3(b) that

$$\begin{aligned}\text{Cov}(Y(t_1), Y(t_2)) &= e^{t_1+t_2}\text{Var}(\int_0^{t_1} e^{-s} dW_s) \\ &= e^{t_1+t_2} \int_0^{t_1} e^{-2s} ds = \frac{1}{2} e^{t_1+t_2}(1 - e^{-2t_1}).\end{aligned}$$