MTH6112 Actuarial Financial Engineering
Coursework Week 6

1. Consider a share with price $S(t)$, $0 \le t \le T$. Suppose that a proportional dividend on this share is paid continuously at rate q and is reinvested into the share. The interest rate compounded continuously is r. Let C be the price of the European call option $\mathrm{Call}(K,T)$ on this share and P be the be the price of the European put option $\mathrm{Put}(K,T)$ on the same share.

   Prove that then the following Call-Put parity relation holds:

   $$C - P = e^{-qT}S(0) - e^{-rT}K.$$

2. Recall the following definition of the index and of its value $I(t)$.

   Definition For n shares with prices $S_1(t)$, $S_2(t), \ldots, S_n(t)$ the index $I(t)$ is defined by

   $$I(t) = \omega_1 S_1(t) + \omega_2 S_2(t) + \cdots + \omega_n S_n(t),$$

   where ω_1, $\omega_2, \ldots, \omega_n$ are positive numbers such that $\sum_{j=1}^{n} \omega_j = 1$. The numbers w_j are called *weights*.

   (a) Suppose that, unlike in the theorem proved in the notes, the strike price K_j for the j^{th} option does depend on j. Moreover, suppose also that the weights ω_j do not necessarily satisfy the relation $\sum_{j=1}^{n} \omega_j = 1$.

   Prove that if $C_j(K_j,t)$ is the price of the call option on the share $S_j(t)$, $j = 1, ..., n$, and $K = \sum_{j=1}^{n} \omega_j K_j$ then

   $$C_I(K,t) \le \sum_{j=1}^{n} \omega_j C_j(K_j,t).$$

   (The notations we use clearly indicate that the expiration time of all options is t.)

   (b) State and prove a similar relation for put options.

3. (a) State the definition of the implied volatility.

(b) Write down the expression for $\frac{\partial C}{\partial \sigma}$. Can you state that this derivative is non-negative?

(c) Prove that implied volatility is uniquely defined (if it exists).

4. As usual, we denote by $W_t \equiv W(t)$ the values at time $t \geq 0$ of the (standard) Wiener process. You are reminded that by definition

$$\int_a^b f(s)dW_s = \lim_{\delta \to 0} \sum_{i=0}^{n-1} f(t_i)\Delta W_i,$$

where

$$a = t_0 < t_1 < ... < t_{n-1} < t_n = b, \;\; \delta = \max_{0 \leq i \leq n-1} t_{i+1} - t_i, \;\text{ and } \Delta W_i = W(t_{i+1}) - W(t_i).$$

(a) Compute the integral $\int_0^3 f(s)dW_s$ in terms of the values of $W(t)$ for a function defined by $f(s) = \begin{cases} -1 & \text{if } 0 \leq s < 1, \\ 1 & \text{if } 1 \leq s < 2, \\ -1 & \text{if } 2 \leq s \leq 3. \end{cases}$

(b) What is the distribution of the integral from part (a)?

(c) Suppose that $f(s) = \begin{cases} 2 & \text{if } 0 \leq s < 1, \\ -2 & \text{if } s \geq 1. \end{cases}$

Compute $Y(t) = \int_0^t f(s)dW_s$ for all $t \geq 0$ (in terms of the values of W).

5. Read the Slides of this week.

a) Find the distribution of the random variables $\int_0^t s^2 dW_s$ and $\int_0^t e^{-s} dW_s$.

b) Compute the variance of the random variables $\int_0^t W_s^2 dW_s$ and $\int_0^t e^{-W(s)} dW_s$.

6. Consider a random process $Y(t),\ t \geq 0$, defined by $Y(t) = \int_0^t f(s)dW_s$.

(a) This process has independent increments. Prove the following particular case of this statement: if $0 < \tau_1 < \tau_2$ then the random variables $Y(\tau_1)$ and $Y(\tau_2) - Y(\tau_1)$ are independent.

Hint. This property is a corollary of the definition of the integral. You have to use the independence of the increments ΔW_i of the Wiener process.

(b) Using the property stated in (a) prove that if $t_1 < t_2$ then

$$\mathrm{Cov}(Y(t_1), Y(t_2)) = \mathrm{Var}(Y(t_1)).$$

(c) Compute $\mathrm{Cov}(Y(t_1), Y(t_2))$ in the case when $Y(t) = \int_0^t e^{t-s} dW_s$.