Actuarial Financial Engineering

Week 6

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9. Call-Put Parity

Theorem 9.1

Suppose that the price $S(t)$ of a share satisfies the relation

$$\tilde{\mathbb{E}}(S(T)) = S_0 e^{rT}, \tag{1}$$

where r is the interest rate compounded continuously and $S_0 = S(0)$. Then

$$C - P = S_0 - e^{-rT} K, \tag{2}$$

9. Call-Put Parity

Proof.

We know that $C = e^{-rT}\tilde{\mathbb{E}}(S(T) - K)^+$ and $P = e^{-rT}\tilde{\mathbb{E}}(K - S(T))^+$. So

$$\begin{aligned} C - P &= e^{-rT}[\tilde{\mathbb{E}}(S(T) - K)^+ - \tilde{\mathbb{E}}(K - S(T))^+] \\ &\overset{(\star)}{=} e^{-rT}\tilde{\mathbb{E}}[(S(T) - K)^+ - (K - S(T))^+] \\ &\overset{(\star\star)}{=} e^{-rT}\tilde{\mathbb{E}}[S(T) - K] \\ &= e^{-rT}(\tilde{\mathbb{E}}(S(T)) - K) \\ &\overset{(\star\star\star)}{=} e^{-rT}(S_0 e^{rT} - K) = S_0 - e^{-rT}K. \end{aligned}$$

This proves the relation $C - P = S_0 - e^{-rT}K$.

It remains to explain the equalities used above:

$(\star)$ is just the standard property of any expectation: $\mathbb{E}X - \mathbb{E}Y = \mathbb{E}(X - Y)$;

$(\star\star)$ is due to $x^+ - (-x)^+ = x$ which holds for any real number x (check this!);

$(\star\star\star)$ is due to condition (1).

9. Call-Put Parity

Definition 9.1

The relation

$$C - P = S_0 - e^{-rT} K.$$

is called the **Call-Put parity** formula.

9. Call-Put Parity

1. Condition (1) is crucial for the proof of the Call-Put Parity formula.
2. Condition (1) is satisfied when $S(t)$ is the price of a share which at time T provides the payoff $S(T)$. This is true for the standard Black-Scholes model. However, this is not true if the share pays **dividends**.
3. *Important exercise.* Prove that if the price $S(t)$ follows the geometric Brownian motion and a dividend is paid continuously at rate q then the following version of the Call-Put Parity formula holds:

$$C - P = e^{-qT} S_0 - e^{-rT} K.$$

   Hint: In this case $\tilde{\mathbb{E}}(S(T)) = e^{(r-q)T} S(0)$. The rest of the proof remains the same.

10. Indices and the diversification of risk
10.1. Definition of an index

Definition 10.1 (Index)

For n shares with prices $S_1(t), S_2(t), \ldots, S_n(t)$ the index $I(t)$ is defined by

$$I(t) = \omega_1 S_1(t) + \omega_2 S_2(t) + \cdots + \omega_n S_n(t),$$

where $\omega_1, \omega_2, \ldots, \omega_n$ are positive numbers such that $\sum_{j=1}^{n} \omega_j = 1$. The numbers $\omega_1, \omega_2, \ldots, \omega_n$ are called weights.

Note that index can be viewed as the price at time t of a portfolio consisting of p_1 shares with price $S_1(t)$, p_2 shares with price $S_2(t)$, ..., p_n shares with price $S_n(t)$.

10.1. Definition of an index

Remarks.

- In the financial world, the expression *a basket of n shares* is used. The weights ω_j specify the number of shares in the basket with the prices $S_j(t)$.
- Typical equity indices are FTSE 100, S&P500, EURO STOXX 50, Nikkei 225.
  The value of these indices is given by a formula like the one above where $S_i(t)$ are the share prices of some stocks called the *constituents* of the index, and the weights ω_i are specified by the organization that produces the index.
  The number of constituent stocks is often added at the end of the name, so for example FTSE 100 involves the price of $n = 100$ shares.

10.1. Definition of an index

Remarks. (cont.)

- Indices are used to achieve *risk diversification*. If we buy a single stock we are exposed to large losses if that stock or business sector under-performs. This is more difficult if we combine different companies as happens in an index.
  To explain this in more precise terms we first recall that for any two random variables X and Y we have:
  $$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + 2\rho_{XY}\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)} + \mathrm{Var}(Y),$$
  where $\rho_{XY} \in [-1,1]$ is the correlation between X and Y. This implies, in the case of negatively correlated random variables, $\rho_{XY} < 0$, that:
  $$\mathrm{Var}(X+Y) < \mathrm{Var}(X) + \mathrm{Var}(Y).$$
  If now $X = S_1$ and $Y = S_2$ are the prices of two negatively correlated assets then the variance of the basket consisting of these two assets is lower than the sum of their variances:
  $$\mathrm{Var}(S_1+S_2) < \mathrm{Var}(S_1) + \mathrm{Var}(S_2), \quad \text{since } \rho_{S_1S_2} < 0.$$

10.2. Options on an index

What is the value of an option on an index?
More precisely, can we estimate the price of an option on $I(t)$ in terms of prices C_j of options on $S_j(t)$?
The answer is given by the following theorem.

10.2. Options on an index

Theorem 10.1

Suppose that $S_1(t), S_2(t), \dots, S_n(t)$ are the prices of the stocks and that the corresponding weights are $\omega_1, \omega_2, \dots, \omega_n$.

Let $C_I(K,t)$ be the price of the call option on the index $I(t)$ and $C_j(K,t)$ be the price of the call options on the stock with the price $S_j(t)$, $j = 1, \dots, n$. Then

$$C_I(K,t) \le \sum_{j=1}^{n} \omega_j C_j(K,t).$$

10.2. Options on an index

Proof. We know that $C_I(K,t) = e^{-rt}\tilde{\mathbb{E}}(I(t)-K)^+$ and $C_j(K,t) = e^{-rt}\tilde{\mathbb{E}}(S_j(t)-K)^+$. Note that

$$I(t) - K = \sum_{j=1}^{n} \omega_j S_j(t) - K = \sum_{j=1}^{n} \omega_j (S_j(t) - K),$$

where we use the equality $K = K\sum_{j=1}^{n}\omega_j = \sum_{j=1}^{n}(\omega_j K)$ (remember that $\sum_{j=1}^{n}\omega_j = 1$). Hence

$$\begin{aligned}(I(t)-K)^+ &= \Big(\sum_{j=1}^{n} \omega_j (S_j(t)-K)\Big)^+ \\ &\leq \sum_{j=1}^{n} (\omega_j (S_j(t)-K))^+ \\ &= \sum_{j=1}^{n} \omega_j (S_j(t)-K)^+\end{aligned}$$

10.2. Options on an index

Proof (cont.).
and therefore

$$e^{-rt}\tilde{\mathbb{E}}(I(t)-K)^{+} \leq e^{-rt}\sum_{j=1}^{n}\omega_j\mathbb{E}(S_j(t)-K)^{+} = \sum_{j=1}^{n}\omega_j e^{-rt}\mathbb{E}(S_j(t)-K)^{+}.$$

We thus proved that

$$C_I(K,t) \leq \sum_{j=1}^{n}\omega_j C_j(K,t).$$

□

10.2. Options on an index

1. In the proof, we use the inequality $(\sum_{j=1}^{n} x_j)^+ \leq \sum_{j=1}^{n}(x_j)^+$.
2. Formally speaking, we use also the fact that all strike prices and strike times are the same. The following exercise shows that the first of these conditions can be relaxed.
   Exercise.
   Suppose that K_j is the strike price of the j^{th} option and $C_j(K_j, t)$ is the price of this call option while the expiry time is t (as before). All numbers K_j may be different but we suppose that $K = \sum_{j=1}^{n} \omega_j K_j$. Prove that then again

$$C_I(K, t) \leq \sum_{j=1}^{n} \omega_j C_j(K_j, t).$$

11. Volatility

We start with the **statement of the problem**.

As you will know, the simplest version of Black-Scholes formula contains 5 parameters of which 4 may be viewed as known. They are:
r - the interest rate (compounded continuously) is defined by the bank,
$S \equiv S(0)$ - the price of the share at time $t = 0$ is what the market tells us,
$K, \; T$ - the strike price and the expiry time of the option, is stated by the contract defining the option.

But what about volatility? Can we find the value of σ?

We shall discuss several solutions to this problem. One should always remember that the Black-Scholes model is an attempt to describe a very complicated phenomenon by relatively simple means.

11. Volatility

Throughout the rest of this section, we suppose that:
1. The price of the share follows the geometric Brownian motion,

$$S(t) = S\, e^{\mu t + \sigma W(t)},$$

where, as usual, $W(t)$ is the standard Wiener process and σ is the volatility parameter that we want to estimate.
2. The price of the Call(K, T) option is given by the Black-Scholes formula

$$C \equiv C(S, K, T, \sigma, r) = S\Phi(\omega) - Ke^{-rT}\Phi(\omega - \sigma\sqrt{T}), \tag{3}$$

where $\omega = \frac{\ln \frac{S}{K} + rT}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}$ and $\Phi(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\omega} e^{-\frac{x^2}{2}} dx$ (as usual).

11.1. Historic volatility

Historic volatility is an estimate of volatility obtained from the values of the prices $S(t)$ recorded during some past period of time.
The word *historic* emphasizes the fact that the estimate is based on the knowledge of the history of the process $S(t)$.

Recall that in statistics, we compute the estimate s^2 of the variance of a random variable X as follows.
1. We record a sequence $x_1, \ldots, x_n$ of independently observed values of the random variable X (called samples).
2. We then calculate the s^2 as follows:

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \overline{x})^2 , \quad \text{where} \quad \overline{x} = \frac{1}{n} \sum_{i=1}^{n} x_i .$$

11.1. Historic volatility

So, to estimate σ we do the following.

- Record the values of $S(t)$ on all working days of, say, a year which precedes the day on which we want to estimate σ. (Usually, such information is readily available from official sources related to the asset traders.)

  More precisely, on the day number j we get the values $S(t_j^{(0)})$ and $S(t_j^{(1)})$, where $t_j^{(0)}$ is the time (of the morning) at which the price is recorded and $t_j^{(1)}$ is the time in the evening.
  Say, $t_j^{(0)} = 9:00$, $t_j^{(1)} = 17:00$. The $\Delta t = t_j^{(1)} - t_j^{(0)}$ should be the same for each day and $j = 1, 2, ..., 251$ (simply because 251 is usually the number of working days in a year).

11.1. Historic volatility

- Set

$$X_j = \ln \frac{S(t_j^{(1)})}{S(t_j^{(0)})} = \ln \frac{S_0 e^{\mu t_j^{(1)} + \sigma W(t_j^{(1)})}}{S_0 e^{\mu t_j^{(0)} + \sigma W(t_j^{(0)})}}$$
$$= \ln e^{\mu \Delta t + \sigma (W(t_j^{(1)}) - W(t_j^{(0)}))}$$
$$= \mu \Delta t + \sigma (W(t_j^{(1)}) - W(t_j^{(0)})).$$

- Then X_j is a sequence of normal independent identically distributed random variables, $X_j \sim \mathcal{N}(\mu \Delta t, \sigma^2 \Delta t)$

11.1. Historic volatility

So, from the statistics formulae above we get:

$$\mu\Delta t \approx \frac{1}{n}\sum_{j=1}^{n} X_j \equiv \bar{X},$$

$$\sigma^2\Delta t \approx \frac{1}{n-1}\sum_{j=1}^{n}(X_j - \bar{X})^2,$$

$$\sigma^2 \approx \frac{1}{\Delta t(n-1)}\sum_{j=1}^{n}(X_j - \bar{X})^2.$$

We thus got an estimate for σ^2 and hence also for

$$\sigma \approx \left(\frac{1}{\Delta t(n-1)}\sum_{j=1}^{n}(X_j - \bar{X})^2\right)^{\frac{1}{2}}.$$

This approximate value of σ is the **historic volatility**.

11.2. Implied volatility

The historic volatility defined in the previous section is useful for computing the prices of options if we are sure that σ does not depend on time (does not change as the time passes by).

Our next estimate of σ does not use the past values of the **prices of the share**.
The idea is as follows.
The market "knows" the **price of the option**: the trade goes on - whether we know σ or not.
So, we can in fact use the market **price of the call option** and view Equation (3) as an equation for σ.
We shall now give a detailed explanation of this idea.

11.2. Implied volatility

Definition 11.1 (Implied volatility)

The implied volatility is the solution σ of the equation

$$C(\sigma) = c_0, \tag{4}$$

where c_0 is the market price of the option at time $t = 0$ and $C(\sigma) \equiv C(S, K, T, \sigma, r)$ is viewed as a function of σ while K, S, T, r are known fixed numbers.

11.2. Implied volatility

Lemma 11.1

Suppose that equation $C(\sigma) = c_0$ has a solution. Then this solution is unique.

Proof.

We know the derivative

$$\nu = \frac{\partial C(\sigma)}{\partial \sigma} = S\sqrt{T} f(\omega) > 0,$$

(where $f(\omega) = \Phi'(\omega) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\omega^2}{2}}$).

We therefore conclude that $C(\sigma)$ is a strictly monotone function of σ and hence it crosses the level c_0 exactly one time. $\square$

11.2. Implied volatility

This simple Lemma allows us to answer the following question: *what are the values of* c_0 *for which the solution to equation* (4) *exists?*
The answer follows from our next lemma.

Lemma 11.2

The following limits exist:

$$\lim_{\sigma\to\infty} C(\sigma) = S \quad \textit{and} \quad \lim_{\sigma\to 0} C(\sigma) = (S - e^{-rT}K)^+.$$

Before proving this Lemma, let us state the answer to the above question.

Corollary 11.1

The (unique) solution σ *to equation* (4) *exists if and only if the price* c_0 *of the option* $Call(K, T)$ *satisfies* $(S - e^{-rT}K)^+ < c_0 < S$.

Exercise Even though this corollary is essentially obvious, do prove it.

11.2. Implied volatility

Proof of Lemma 11.2.

The existence of the limits follows from the monotonicity of the function $C(\sigma)$.

To compute the first limit, recall that $\omega = \frac{\log \frac{S}{K} + rT}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}$ an hence $\omega \to \infty$ when $\sigma \to \infty$. Also

$$\omega - \sigma\sqrt{T} = \frac{\log \frac{S}{K} + rT}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T} \to -\infty \quad \text{as} \quad \sigma \to \infty.$$

Taking into account that $\Phi(\infty) = 1$ and $\Phi(-\infty) = 0$ we obtain from the Black-Scholes formula that

$$\lim_{\sigma\to\infty} C(\sigma) = \lim_{\sigma\to\infty} \left(S\Phi(\omega) - Ke^{-rT}\Phi(\omega - \frac{1}{2}\sigma\sqrt{T}) \right) = S\Phi(\infty) - Ke^{-rT}\Phi(-\infty) = S.$$

11.2. Implied volatility

Proof of Lemma 11.2 (cont).

Next, let us find $\lim_{\sigma \to 0} C(\sigma)$. To do that, we recall the other expression for $C(\sigma)$:

$$C(\sigma) = e^{-rT}\mathbb{E}(Se^{\tilde{\mu}T + \sigma W(T)} - K)^+.$$

Hence, when $\sigma = 0$ we obtain

$$C(0) = e^{-rT}\mathbb{E}(Se^{\tilde{\mu}T} - K)^+ = e^{-rT}(Se^{\tilde{\mu}T} - K)^+.$$

Since $\tilde{\mu} = r - \frac{1}{2}\sigma^2$, we get when $\sigma = 0$,

$$C(0) = e^{-rT}(Se^{rT} - K)^+ = (S - e^{-rT}K)^+.$$

□

Remark.

The first limit computed above shows that the price of a European call option is always lower than the price of the underlying share: $C < S$. You are supposed to know that from FMI.

11.3. The volatility smile

If we suppose that the Black-Scholes model provides an exact description of the real world behaviour of the prices $S(t)$ and $C(K, T, S, \sigma, r)$, then the solution σ to equation (4) should not depend on the value of K or T (since all of them are independent parameters of the model, and σ objectively exists independent on the options).

This claim is easy to check experimentally because traders sell different European call options on the same asset. Obviously, any two such option either have different K's or different T's or differ in both K and T.

11.3. The volatility smile

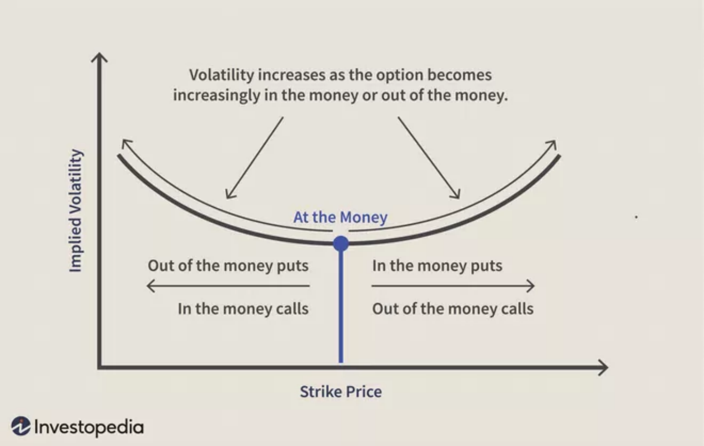

Figure: This is an example of the real life volatility smile. Source: Investopedia, By Cory Mitchell

11.3. The volatility smile

Figure 1 presents the graph of volatilities of an option.
This kind of behaviour of the volatility σ as a function of K (while all other parameters are fixed) has a remarkable name - it is called *the volatility smile*.

Since one can see a clear non-trivial dependence of σ on K, the conclusion is that the Black-Scholes model does not provide an exact description of the real world prices! Nevertheless, even the imperfect description it provides is enormously important.

12. Stochastic Calculus

12.1. The standard definition of the integral

Let us recall the standard definition of the integral $\int_a^b f(x)dx$, where $f : [a, b] \to$ is a real valued function on $[a, b]$. To define the integral we need the following construction.

1. Choose any $n-1$ (interior) points from $[a, b]$ such that
$$a = x_0 < x_1 < \ldots, x_{n-1} < x_n = b.$$
2. Consider the integral sum $\sum_{i=0}^{n-1} f(\xi_i)\Delta x_i$, where $\Delta x_i = x_{i+1} - x_i$ and $\xi_i \in [x_i, x_{i+1}]$ (and is arbitrary otherwise).
3. Set $\delta = \max_{0 \le i \le n-1} \Delta x_i$.
4. Finally, the integral of the function $f(x)$ over $[a, b]$ is, by definition, the limit of the integral sum (if this limit exists):
$$\int_a^b f(x)dx = \lim_{\delta \to 0} \sum_{i=0}^{n-1} f(\xi_i)\Delta x_i$$

12.1. The standard definition of the integral

Theorem 12.1

If $f(x)$ is a continuous function on $[a, b]$,
then the limit $\lim_{\delta \to 0} \sum_{i=1}^{n-1} f(\xi_i)\Delta x_i$ exists
(and does not depend on the choice of x_i, ξ_i, $0 \le i \le n$).

Before we define the stochastic integral (also called the Ito integral) we have to recall several properties of normal random variables and of the Wiener process which will play a very important role in the study of these integrals.

12.2. Stochastic integrals (the Ito integrals)

Properties of normal random variables and of the Wiener process.

1. If $Z_1, Z_2, \ldots, Z_n$ are independent normal random variables, $Z_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$, then

$$\sum_{i=1}^{n} Z_i \sim \mathcal{N}\Big(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\Big). \tag{5}$$

2. As usual, we denote by $W(t) \equiv W_t$ the standard Wiener process. By the definition of the Wiener process the following properties hold:
   1. $W(0) = 0$.
   2. $W(t+s) - W(t) \sim \mathcal{N}(0, s)$ if $s > 0$.
   3. Let $t_0 = 0 < t_1 < t_2 < \ldots \cdots < t_{n-1} < t_n = t$ be any points from the interval $[0,\ t]$. Set

$$\Delta W_i = W(t_{i+1}) - W(t_i),\ \ i = 0, 1 \ldots, n-1 \ \text{ and } \ \Delta t_i = t_{i+1} - t_i. \tag{6}$$

   Then ΔW_i, $i = 0,\ 1,\ \ldots,\ n-1$ are independent normal random variables, $\Delta W_i \sim \mathcal{N}(0, \Delta t_i)$.

12.2. Stochastic integrals (the Ito integrals)

Our goal is to define

1. $\int_0^t f(s)dW_s$, where $f(s)$ is a "usual" function (not random).
   This is a relatively simple case of a stochastic integral.
2. $\int_0^t f(W_s)dW_s$ - the stochastic integral of a function of a Wiener process which is a somewhat more complicated case.

(a) Stochastic integral $\int_0^t f(s)dW_s$

(b) Distribution of the random variable $\int_0^t f(s)dW_s$

(c) Stochastic integral $\int_0^t f(W_s)dW_s$

12.2. Stochastic integrals (the Ito integrals)

(a) Stochastic integral $\int_0^t f(s)dW_s$

Definition 12.1

Let $t_0 = 0 < t_1 < t_2 < \cdots < t_n = t$ be a sequence of points in $[0, t]$ and define $\delta = \max_i \Delta t_i$. Then

$$\int_0^t f(s)dW_s = \lim_{\delta \to 0} \sum_{i=0}^{n-1} f(t_i)\Delta W_i \tag{7}$$

if this limit exists.

Theorem 12.2

If $f(x)$ is differentiable and $f'(x)$ is a continuous function then the limit in (7) exists.

12.2. Stochastic integrals (the Ito integrals)

Let us consider several simple examples.

Example 1.

$f(x) = c$ (constant), then

$$\int_0^t c dW_s = \lim_{\max_i \Delta t_i \to 0} \sum_{i=0}^{n-1} c \Delta W_i = c \lim_{\max_i \Delta t_i \to 0} \sum_{i=0}^{n-1} \Delta W_i$$

where as above $\Delta W_i = W(t_{i+1}) - W(t_i)$. Since

$$\begin{aligned}\sum_{i=0}^{n-1} \Delta W_i &= (W(t_1) - W(t_0)) + (W(t_2) - W(t_1)) + \cdots + (W(t_n) - W(t_{n-1})) \\ &= W(t_n) - W(t_0) = W(t)\end{aligned}$$

we see that $\lim_{\max_i \Delta t_i \to 0} \sum_{i=0}^{n-1} \Delta W_i = \sum_{i=0}^{n-1} \Delta W_i = W(t)$ and therefore

$$\int_0^t c dW_s = cW(t).$$

12.2. Stochastic integrals (the Ito integrals)

Remark.
Always remember the identity

$$\sum_{i=0}^{n-1}(b_{i+1} - b_i) = (b_1 - b_0) + (b_2 - b_1) + \cdots + (b_n - b_{n-1}) = b_n - b_0.$$

We use it in the above example with $b_i = W(t_i)$.

12.2. Stochastic integrals (the Ito integrals)

Example 2.

$$f(x) = \begin{cases} 1, & 0 \le x < 1.5, \\ -1, & 1.5 \le x \le 2. \end{cases}$$

Then

$$\begin{aligned} \int_0^2 f(s)dW_s &= \int_0^{1.5} f(s)dW_s + \int_{1.5}^2 f(s)dW_s \\ &= \int_0^{1.5} dW_s - \int_{1.5}^2 dW_s = W(1.5) - (W(2) - W(1.5)) \\ &= 2W(1.5) - W(2) \end{aligned}$$

We use $\int_a^b dW_s = W(b) - W(a)$.

12.2. Stochastic integrals (the Ito integrals)

Question What is the distribution of this integral? Denote
$Y \equiv W(1.5) - (W(2) - W(1.5))$.
Answer Since $W(1.5) \sim \mathcal{N}(0, 1.5)$, $W(2) - W(1.5) \sim \mathcal{N}(0, 0.5)$ and these random variablesare *independent*, their difference $Y \sim \mathcal{N}(0, 2)$. (This is a particular case of (5). Explain this statement.)

Exercise.

$$f(x) = \begin{cases} 1, & 0 \leq x < 1, \\ 2, & 1 \leq x < 1.5, \\ -1.5, & 1.5 \leq x \leq 3. \end{cases}$$

What is the distribution of $\int_0^3 f(s) dW_s$?

12.2. Stochastic integrals (the Ito integrals)

(b) Distribution of the random variable $\int_0^t f(s)dW_s$

The integral $\int_0^t f(s)dW_s$ is a random variable because it is defined as a limit of a sum of random variables.

Question What is the distribution of this random variable?

It is remarkable that this question has a simple answer. Namely, our next theorem states that this random variable has a **normal** distribution and, moreover, it is relatively easy to compute the parameters of this distribution.
We shall see later that this fact plays a very important role in constructing solutions to some questions arising in financial mathematics.

12.2. Stochastic integrals (the Ito integrals)

Theorem 12.3

$\int_0^t f(s)dW_s \sim \mathcal{N}\left(0, \int_0^t (f(s))^2 ds\right)$.

Proof.

By the definition of a limit,

$$\int_0^t f(s)dW_s \simeq \sum_{i=0}^{n-1} f(t_i)\Delta W_i.$$

Since ΔW_i are independent random variables and $\Delta W_i = W(t_{i+1}) - W(t_i) \sim \mathcal{N}(0, \Delta t_i)$, the random variables $f(t_i)\Delta W_i$ are also independent and $f(t_i)\Delta W_i \sim \mathcal{N}(0, f(t_i)^2 \Delta t_i)$. (Note that the last statement makes use of the fact that $f(t_i)$ are not random variables!)

12.2. Stochastic integrals (the Ito integrals)

Proof (cont).
Next, due to property (5) we conclude that

$$\sum_{i=0}^{n-1} f(t_i)\Delta W_i \sim \sum_{i=0}^{n-1} \mathcal{N}(0, f(t_i)^2 \Delta t_i) = \mathcal{N}(0, \sum_{i=0}^{n-1} f(t_i)^2 \Delta t_i).$$

By Theorem 12.1,

$$\lim_{\max_i \Delta t_i \to 0} \sum_{i=0}^{n-1} f(t_i)^2 \Delta t_i = \int_0^t f(s)^2 ds$$

which finishes the proof.□
Exercise. Find the distributions of the random variables defined in the examples of (a).

12.2. Stochastic integrals (the Ito integrals)

(c) Stochastic integral $\int_0^t f(W_s)dW_s$

As before, let $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = t$ and $\delta = \max_{0\le i\le n-1}(t_{i+1} - t_i)$.

Definition 12.2

Let $f :\to$ be a function. If the limit $\lim_{\delta\to 0}\sum_{i=0}^{n-1} f(W(t_i))\Delta W_i$ exists, then we say that

$$\int_a^b f(W_t)dW_t = \lim_{\delta\to 0}\sum_{i=0}^{n-1} f(W(t_i))\Delta W_i. \tag{8}$$

Theorem 12.4

Suppose that the function $f :\to$ has a bounded continuous derivative $f'(x)$. Then the limit in (8) exists.

12.2. Stochastic integrals (the Ito integrals)

The just defined integral is of course again a random variable. However, it may be very difficult to finding the distribution of this random variable.
We finish this section by stating two properties of this stochastic integral:

Theorem 12.5

$$\mathbb{E}\left(\int_a^b f(W_t)dW_t\right) = 0, \tag{9}$$

$$\mathrm{Var}\left(\int_a^b f(W_t)dW_t\right) = \int_a^b \mathbb{E}[(f(W_t)^2]dt \tag{10}$$

12.2. Stochastic integrals (the Ito integrals)

Explanation (not examinable.)
First, let us introduce notations which will make our calculation less cumbersome. We set $W_i \equiv W(t_i)$, $f_i \equiv f(W_i)$, $\Delta W_i \equiv W(t_{i+1}) - W(t_i)$.
Since W_i and ΔW_i are independent, also the random variables $f_i = f(W(t_i))$ and ΔW_i are independent. Hence

$$\mathbb{E}(f_i \Delta W_i) = \mathbb{E}(f_i) \times \mathbb{E}(\Delta W_i) = 0 \text{ because } \mathbb{E}(\Delta W_i) = 0. \tag{11}$$

It is now obvious that

$$\mathbb{E}\left(\sum_{i=0}^{n-1} f_i \Delta W_i\right) = \sum_{i=0}^{n-1} \mathbb{E}\left(f_i \Delta W_i\right) = 0$$

and (9) follows because $\mathbb{E}\left(\int_a^b f(W_t)dW_t\right) = \lim_{\delta \to 0} \mathbb{E}\left(\sum_{i=0}^{n-1} f(W(t_i))\Delta W_i\right)$.

12.2. Stochastic integrals (the Ito integrals)

Explanation (cont).

To explain (10), note that if $i < j$ then

$$\mathrm{Cov}(f_i \Delta W_i, f_j \Delta W_j) = \mathbb{E}[f_i \Delta W_i \times f_j \Delta W_j] = \mathbb{E}[f_i \Delta W_i f_j] \times \mathbb{E}(\Delta W_j) = 0$$

where the expectation factorizes because ΔW_j is independent of the other three random variables. We thus have that

$$\mathrm{Var}\left(\sum_{i=0}^{n-1} f(W_i)\Delta W_i\right) = \sum_{i=0}^{n-1} \mathrm{Var}(f_i \Delta W_i) \tag{12}$$

$$= \sum_{i=0}^{n-1} \mathbb{E}(f_i^2 \Delta W_i^2) = \sum_{i=0}^{n-1} \mathbb{E}(f_i^2) \times \mathbb{E}(\Delta W_i^2) \tag{13}$$

$$= \sum_{i=0}^{n-1} \mathbb{E}(f_i^2) \times \Delta t_i \tag{14}$$

12.2. Stochastic integrals (the Ito integrals)

Explanation (cont).
The last sum converges, as $\delta \to 0$, to $\int_a^b \mathbb{E}[(f(W_t)^2]dt$ and this implies (10) because $\mathrm{Var}\left(\int_a^b f(W_t)dW_t\right) = \lim_{\delta \to 0} \mathrm{Var}\left(\sum_{i=0}^{n-1} f(W(t_i))\Delta W_i\right)$.

Remark.
1. In the above computation, we use $\mathbb{E}[\Delta W_i^2] = t_{i+1} - t_i = \Delta t_i$. (Lemma 1.2, Week 1)
2. We use the following fact which you are supposed to know from second year probability courses: if $X_1, ..., X_n$ are such that $\mathrm{Cov}(X_i, X_j) = 0$ when $i \neq j$ then

$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}(X_i).$$