MTH6112 Actuarial Financial Engineering
Coursework Week 5

You may need the following theorems to solve the questions.

Theorem 1 *Suppose that conditions (i), (ii), (iii) of Theorem 1 in Coursework Week 4 are satisfied and in addition dividend is paid continuously at rate q and is reinvested in the underlying asset.*

Then the price $C_q(S,t)$ of the derivative with payoff (exercise) time t is given by

$$C_q(S,t) = e^{-rt}\mathbb{E}\left(R(\tilde{S}(t))\right), \quad \text{where} \quad \tilde{S}(t) = Se^{\tilde{\mu}t+\sigma W(t)} \quad \text{and } \tilde{\mu} = r - q - \frac{\sigma^2}{2}.$$

Corollary We could equivalently say that $C_q(S,t) = C(e^{-tq}S,t)$.

Theorem 2 *Suppose that conditions (i), (ii), (iii) of Theorem 1 in Coursework Week 4 are satisfied and in addition the proportionate dividend $D = dS(t_0)$ is payed at time t_0.*

Then the price $C_2(S,t,d)$ of the derivative exercised at time t is computed as follows:

(a)If $t \le t_0$ then

$$C_2(S,t,d) = e^{-rt}\mathbb{E}\left(R(\tilde{S}(t))\right), \quad \text{where} \quad \tilde{S}(t) = Se^{\tilde{\mu}t+\sigma W(t)} \quad \text{and } \tilde{\mu} = r - \frac{\sigma^2}{2}.$$

(b)If $t > t_0$ then

$$C_2(S,t,d) = e^{-rt}\mathbb{E}\left(R(\tilde{S}(t))\right), \quad \text{where} \quad \tilde{S}(t) = (1-d)Se^{\tilde{\mu}t+\sigma W(t)} \quad \text{and } \tilde{\mu} = r - \frac{\sigma^2}{2}.$$

Remark Equivalently, $C_2(S,t,d) = C(S,t)$ if $t \le t_0$ and $C_2(S,t,d) = C((1-d)S,t)$ if $t > t_0$.

1. Recall the Question 2 of Coursework Week 4. Suppose that the price $S(t)$ of a share is described by the GBM with parameters S, μ, σ, r.

   Suppose now that the above share provides a dividend yield of rate q which is paid continuously and is reinvested in the share. What is the price C of the derivative with the same payoff function?

Solution By Theorem 1, the only difference with Q2 of last week is that $\tilde{\mu} = r - q - \frac{\sigma^2}{2}$. So once again

$$C = e^{-rt}\mathbb{E}\left(R(\tilde{S}(t))\right) = e^{-rT}\mathbb{E}\left(\tilde{S}(t)^2 + 1\right) = S^2 e^{-rT+2\tilde{\mu}T}\mathbb{E}(e^{2\sigma W(t)}) + e^{-rT}$$

but this time

$$C = S^2 e^{(r-2q+\sigma^2)T} + e^{-rT}.$$

2. Recall the Question 3 of Coursework Week 4. Suppose again that the price $S(t)$ of a share is described by the GBM with parameters $S,\ \mu,\ \sigma,\ r$.

   Consider an option with expiration time T and payoff function given by

$$R(S(T)) = \begin{cases} K & \text{if } S(T) < K, \\ 0 & \text{if } S(T) \geq K. \end{cases}$$

   (Note that if a portfolio consists of 1 share and 1 such option then the payoff of at least £K is guaranteed.)

   (a) Suppose now that the above share provides a dividend yield of rate q which is paid continuously and is reinvested in the share. What is the price C_q of the derivative with the same payoff function?

   Solution By Theorem 1 and its corollary,

$$C_q(S,T) = C(Se^{-qT}, T) = e^{-rT}K\Phi(b(Se^{-qT,T})).$$

   (b) Suppose that a discrete proportionate dividend of rate d is paid at time $T/2$ and is immediately reinvested in the share. The expiration time of the option is t, $0 < t \leq T$. Write down the formulae for the price of this option in the following 2 cases: $t \leq T/2$ and $T/2 < t \leq T$.

   Solution By Theorem 2, $C_2(S,t,d) = C(S,t) = e^{-rt}K\Phi(b(S,t))$ if $t \leq T/2$ and $C_2(S,t,d) = C(S,t) = e^{-rt}K\Phi(b((1-d)S,t))$ if $t > T/2$.

3. Suppose that the price of a share is $S(t)$, $0 \leq t \leq T$. Suppose also that a discrete proportional dividend is paid at time t_0 at rate d. Prove that if $S(t) > (1-d)S(t_0)$ for all $t \in (t_0, t_0 + \epsilon)$, where $\epsilon > 0$, then there is an arbitrage opportunity.

   Solution If $S(t) > (1-d)S(t_0)$ for all $t \in (t_0, t_0+\epsilon)$ then the arbitrage is achieved as follows.

   1. At time $t = t_0$ borrow $S(t_0)$ from a bank and buy the share for $S(t_0)$. As the owner of the share, you get the dividend $d\,S(t_0)$.

2. At time $\bar{t}$, $t_0 < \bar{t} < t_0 + \epsilon$ sell the share for $S(\bar{t})$ and give $S(t_0)$ back to the bank.

Your return now is

$$d\,S(t_0) + S(\bar{t}) - S(t_0) = S(\bar{t}) - (1-d)S(t_0) > 0.$$

You thus have not invested any of your money but got a positive return (and that is what arbitrage is). □

Remark This solution doesn't take into account the fact that you are supposed to give back to the bank $\mathrm{e}^{r(\bar{t}-t_0)}S(t_0)$ which is more than $S(t_0)$. This does not matter because $\bar{t} - t_0$ is a very small time interval and therefore the difference $\mathrm{e}^{r(\bar{t}-t_0)} - 1$ is so small that it can be ignored.

4. Suppose that the price $S(t)$ of the share is driven by a geometric Brownian motion with parameters S, μ, σ, that is $S(t) = Se^{\mu t + \sigma W(t)}$. Suppose also that a proportional dividend on this share is paid continuously at rate $q > 0$ and is reinvested in the share. The continuously compounded interest rate is r. Compute the no-arbitrage price of a derivative with the payoff function $R(T) = \frac{1}{T}\int_0^T S(t)dt$.

   Solution

   By the general theorem (Theorem 5.6) and by the definition of $R(T)$,

$$C = e^{-rT}\tilde{E}\left(\frac{1}{T}\int_0^T S(t)dt\right) = \frac{e^{-rT}}{T}\tilde{E}\left(\int_0^T S(t)dt\right). \tag{1}$$

   By the general rule, in order to compute $\tilde{E}$ we have to replace $S(t)$ in the expression for $R(T)$ by $\tilde{S}(t)$ and then compute the usual expectation. Thus

$$\tilde{E}\left(\int_0^T S(t)dt\right) = E\left(\int_0^T \tilde{S}(t)dt\right).$$

   As stated in lectures, it is possible to change the order of the two operations:

$$E\left(\int_0^T \tilde{S}(t)dt\right) = \int_0^T E\left(\tilde{S}(t)\right)dt.$$

   Next, we know that

$$E\left(\tilde{S}(t)\right) = E\left(Se^{\tilde{\mu}t+\sigma W_t}\right) = Se^{\tilde{\mu}t+\frac{\sigma^2}{2}t}.$$

Since $\tilde{\mu} + \frac{\sigma^2}{2} = r - q$, we have

$$E\left(\tilde{S}(t)\right) = Se^{(r-q)t}$$

and we obtain

$$E\left(\int_0^T \tilde{S}(t)dt\right) = \int_0^T Se^{(r-q)t}dt = \frac{S}{r-q}(e^{(r-q)T} - 1).$$

Finally we obtain from (1):

$$C = \frac{e^{-rT}S}{(r-q)T}(e^{(r-q)T} - 1) = \frac{S}{(r-q)T}(e^{-qT} - e^{-rT}).$$

5. A company's share is currently traded at the price of £18.49. A dividend on this share is paid continuously at rate q and is reinvested in the share. Two options are available on the market with the same strike price of £18 and the same maturity time of 6 months. European Call option is worth of £1.72 and a European Put option is priced at £1.52. Assuming the continuously compounded interest rate is 16%, find the dividend rate.
   Hint: The Call-Put parity for a dividend paying share is

   $$C - P = Se^{-qT} - Ke^{-rT}.$$

   Please compare it with the Call-Put parity you learnt in Financial Mathematics 1. We will further discuss it in Week 6.
   Solution: Using the Call-Put parity for a dividend paying share, where $C = 1.72$, $P = 1.52$, $S = 18.49$, $T = 0.5$, $K = 18$, $r = 0.16$. Thus

   $$q = -\frac{1}{T}\ln\frac{C - P + Ke^{-rT}}{S} = 0.1897 (= 18.97\%).$$