MTH6112 Actuarial Financial Engineering
Coursework Week 4

You may need the following theorem to solve the questions.

Theorem 1 *Suppose that:*

(i) The price of an asset is driven by a GBM with parameters S, μ, σ, *that is* $S(t) = Se^{\mu t+\sigma W(t)}$.

(ii) The interest rate compounded continuously is r.

(iii) A derivative (on the asset) has a payoff function $R(S(t))$ *with payoff time* t.

Then the no-arbitrage price of this derivative is given by

$$C(S,t) = e^{-rt}\mathbb{E}\left(R(\tilde{S}(t))\right), \quad \textit{where} \;\; \tilde{S}(t) = Se^{\tilde{\mu}t+\sigma W(t)} \;\; \textit{and} \; \tilde{\mu} = r - \frac{\sigma^2}{2}.$$

Remark The price of the derivative depends on all parameters. However, we deliberately emphasize the dependence on two parameters, the initial price S of the asset and the time t at which the option expires / is exercised.

1. The prices of the stock of F. Bancroft & Sons follow a geometric Brownian motion with parameters $\mu = 0.15$ and $\sigma = 0.21$. Presently, the stock's price is 38 pounds. Consider a call option having three months until its expiration time and having a strike price of 41 pounds.

   (a) What is the probability that the call option will be exercised?

   (b) If the interest rate is 5%, what is the Black-Scholes price of the call?

   Solution Let $S(t)$ denote the price of the Bancroft Stock at time t, where t is measured in years. We are told that $S(t)$ is geometric Brownian motion with drift parameter $\mu = 0.15$, volatility parameter $\sigma = 0.21$ and starting parameter $S = 38$, that is

   $$S(t) = S\exp(\mu t + \sigma W(t)),$$

   where $W(t)$ denotes the Wiener process. Let $K = 41$ denote the strike price and let $t = 1/4$ denote the expiration time of the call option.

(a) The option will be exercised if $S(t) > K$. Thus, the desired probability is

$$\mathbb{P}(S(t) > K) = \mathbb{P}(S\exp(\mu t + \sigma W(t)) > K) = \mathbb{P}\left(\frac{W(t)}{\sqrt{t}} > \frac{\log\frac{K}{S} - \mu t}{\sigma\sqrt{t}}\right)$$
$$= \mathbb{P}\left(\frac{W(t)}{\sqrt{t}} > 0.37\right) = 1 - \Phi(0.37) = 1 - 0.6443 = 0.3557\,.$$

Here we have used the fact that

$$\frac{W(t)}{\sqrt{t}} \sim \mathrm{N}(0,1)\,.$$

Thus the probability that the call option will be exercised is 36%.

(b) Let $r = 0.05$ denote the nominal interest rate. The Black-Scholes price C of the call is given by the Black-Scholes Formula

$$C = S\Phi(\omega) - Ke^{-rt}\Phi(\omega - \sigma\sqrt{t})\,,$$

where

$$\omega = \frac{rt + \frac{\sigma^2 t}{2} - \log\frac{K}{S}}{\sigma\sqrt{t}}\,.$$

Now

$$\omega = -0.552127 \quad \text{and} \quad \omega - \sigma\sqrt{t} = -0.657127\,,$$

so

$$C = 38\Phi(-0.55) - 41e^{-0.05/4}\Phi(-0.66) = 38(1-\Phi(0.55)) - 41e^{-0.05/4}(1-\Phi(0.66))$$
$$= 38(1 - 0.7088) - 41e^{-0.05/4}(1 - 0.7454) = 0.65\,.$$

Thus, the Black-Scholes price of the call option is 65 pence.

2. Suppose that the price $S(t)$ of a share is described by the GBM with parameters $S,\ \mu,\ \sigma,\ r$.

Consider a derivative on this share which provides at time T a payoff $R(S(T)) = S(T)^2 + 1$. Compute the no-arbitrage price C of this derivative.

Remark. This is an example of an artificial 'option'. The advantage of this example is that the answer can be expressed explicitly in terms of the parameters of the model.

Solution By Theorem 1,

$$C = e^{-rt}\mathbb{E}\left(R(\tilde{S}(t))\right) = e^{-rT}\mathbb{E}\left(\tilde{S}(t)^2 + 1\right) = S^2 e^{-rT+2\tilde{\mu}T}\mathbb{E}(e^{2\sigma W(t)}) + e^{-rT}.$$

We have shown before (see CW1) that $\mathbb{E}(e^{m\sigma W(t)}) = e^{m^2\sigma^2 t/2}$. Hence, taking into account the expression for $\tilde{\mu}$ we obtain:

$$C = S^2 e^{rT+\sigma^2 T} + e^{-rT}.$$

3. Suppose again that the price $S(t)$ of a share is described by the GBM with parameters $S,\ \mu,\ \sigma,\ r$.

   Consider now an option with expiration time T and payoff function given by

$$R(S(T)) = \begin{cases} K & \text{if } \ S(T) < K, \\ 0 & \text{if } \ S(T) \geq K. \end{cases}$$

   (Note that if a portfolio consists of 1 share and 1 such option then the payoff of at least £K is guaranteed.)

   (a) Consider the case when no dividend is paid. Compute the no-arbitrage price of this option.

   We present two solutions to this part of the problem

   Solution 1. By the general Theorem (Theorem 5.3, Week 3-4)

$$C(S,T) = e^{-rT}\mathbb{E}\left(R(\tilde{S}(T))\right). \tag{1}$$

   The $R(S(T))$ can be viewed as a random variable taking value K if $R(S(T)) < K$ and 0 otherwise. Hence its expectation with respect to the risk-neutral probability is

$$\mathbb{E}\left(R(\tilde{S}(T))\right) = K\tilde{\mathbb{P}}(S(T) < K) = K\mathbb{P}(\tilde{S}(T) < K).$$

   The inequality $\tilde{S}(T) < K$ can be rewritten as $Se^{\tilde{\mu}T+\sigma\sqrt{T}Z} < K$, where $W(T) = \sqrt{T}Z$ with $Z \sim \mathcal{N}(0,1)$. The last inequality is equivalent to

$$Z < (\ln\frac{K}{S} - \tilde{\mu}T)/(\sigma\sqrt{T}) = \frac{\ln\frac{K}{S} - rT + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}} \equiv b(S,T).$$

   Therefore $\mathbb{P}(\tilde{S}(T) < K) = \Phi(b(S,T))$, where $\Phi(b)$ is the cumulative distribution function of the the standard normal random variable. (This result will be used in your Assessed Coursework).
   Finally, the price

$$C(S,T) = e^{-rT}K\Phi(b(S,T)).$$

Solution 2. By Theorem 1,

$$C(S,T) = e^{-rT}\mathbb{E}\left(R(\tilde{S}(T))\right) = e^{-rT}\int_{-\infty}^{\infty} R(Se^{\tilde{\mu}T+\sigma\sqrt{T}x})f(x)dx, \quad (2)$$

where $f(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ is the standard normal density function (see remark below). By the definition of the function R, it takes value K when $Se^{\tilde{\mu}T+\sigma\sqrt{T}x} < K$ and is 0 otherwise. Solving the inequality, we see that $R(...) < K$ iff

$$x < (\ln\frac{K}{S} - \tilde{\mu}T)/(\sigma\sqrt{T}) = \frac{\ln\frac{K}{S} - rT + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}} \equiv b(S,T).$$

So

$$C(S,T) = e^{-rT}\int_{-\infty}^{\infty} R(Se^{\tilde{\mu}T+\sigma\sqrt{T}x})f(x)dx = e^{-rT}\int_{-\infty}^{b(S)} Kf(x)dx = e^{-rT}K\Phi(b(S,T)),$$

where $\Phi(b)$ is the cumulative distribution function of the the standard normal random variable.

Remark In (3), we use the following fact: if $g(z)$ is a 'good' function then (for any $t > 0$

$$\mathbb{E}\left(g(W(t))\right) = \int_{-\infty}^{\infty} g(x)f_{W(t)}(x)dx = \int_{-\infty}^{\infty} g(\sqrt{t}x)f(x)dx,$$

where $f_{W(t)}(x) = \frac{1}{\sqrt{2\pi t}}e^{\frac{x^2}{2t}}$ is the pdf of $W(t)$ and $f(x)$ is as above.

(b) (This question is optional and will not be examined.) Consider again the case when no dividend is paid and the expiration time is T.

If you are the seller of this option, what should be your hedging strategy? Namely, how many shares must be in your portfolio and how much money should be deposited in the bank at time t, $0 \le t \le T$, in order for you to be able to meet your obligation at time T?

Solution The total cost of the hedging portfolio at time t should be $C(S_t, T-t)$ and the number of shares in the portfolio should be $\Delta(t) = \left.\frac{\partial C(S,T-t)}{\partial S}\right|_{S=S_t}$. (Please refer to Slide 18 - 25, Week 5).

Here and below, we write S_t for $S(t)$ to simplify the notation.

Using the formula for $C(S,T)$ obtained above, we can write

$$C(S_t, T-t) = e^{-r(T-t)}K\Phi(b(S_t, T-t)).$$

Hence, the number of shares that must be in the portfolio is

$$\Delta(t) = e^{-rT} K \frac{\partial \Phi(b(S_t, T-t))}{\partial S_t} = e^{-r(T-t)} K f(b(S_t, T-t)) \frac{\partial b(S_t, T-t)}{\partial S_t}$$
$$= \frac{-1}{S_t \sigma \sqrt{T-t}} e^{-r(T-t)} K f(b(S_t, T-t)).$$

This can be written in a more explicit form:

$$\Delta(t) = \frac{-K}{S_t \sigma \sqrt{2\pi(T-t)}} e^{-r(T-t) - \frac{(\ln \frac{K}{S_t} - r(T-t) + \frac{\sigma^2}{2}(T-t))^2}{2\sigma^2(T-t)}}.$$

Finally, the amount that should be deposited in the bank is

$$C(S_t, T-t) - \Delta(t) S(t) = e^{-r(T-t)} K \left[\Phi(b(S_t, T-t)) + \frac{1}{\sigma\sqrt{T-t}} f(b(S_t, T-t)) \right].$$

4. Compute $\mathbb{E}\left(e^{aW(t)+bW(t+s)}\right)$, where $t > 0$ and $s > 0$.

   Solution Denote $Y = aW(t) + bW(t+s)$. Notice that

$$Y = (a+b)W(t) + b(W(t+s) - W(t)).$$

   So, Y is a sum of 2 independent random variables and hence $e^Y = e^{(a+b)W(t)} \times e^{b(W(t+s)-W(t))}$ is a product of two independent random variables. It follows that

$$\mathbb{E}\left(e^{aW(t)+bW(t+s)}\right) = \mathbb{E}\left[e^{(a+b)W(t)} \times e^{b(W(t+s)-W(t))}\right] = \mathbb{E}\left[e^{(a+b)W(t)}\right] \times \mathbb{E}\left[e^{b(W(t+s)-W(t))}\right].$$

   We know that $\mathbb{E}\left(e^{\sigma W(t)}\right) = e^{\frac{\sigma^2}{2}t}$ (see Week 1). Since $W(t+s) - W(t) \sim \mathcal{N}(0, s)$ and therefore

$$\mathbb{E}\left[e^{b(W(t+s)-W(t))}\right] = \mathbb{E}\left[e^{bW(s)}\right] = e^{\frac{b^2}{2}s}.$$

   Hence

$$\mathbb{E}\left(e^{aW(t)+bW(t+s)}\right) = e^{\frac{(a+b)^2}{2}t} \times e^{\frac{b^2}{2}s}. \tag{3}$$

5. The price of a share $S(t)$ evolves according to a Geometric Brownian Motion with parameters S, μ, σ. The continuously compounded interest rate is r. A derivative on this option has the payoff function

$$R(T) = \frac{1}{T} \int_0^T S(t) S(T) dt.$$

The payoff time is T. What is the no-arbitrage price of this derivative?

Hint Use the result obtained in the former question.

Solution

By the general theorem (see Theorem 5.6 in typeset Notes 3),

$$C = e^{-rT}\tilde{\mathbb{E}}\left(\frac{1}{T}\int_0^T S(t)S(T)dt\right) = \frac{e^{-rT}}{T}\tilde{\mathbb{E}}\left(\int_0^T S(t)S(T)dt\right). \tag{4}$$

To compute this expectation over the risk-neutral probability, we have to turn $\tilde{\mathbb{E}}$ into $\mathbb{E}$ by replacing $S(t)$ and $S(T)$ by $\tilde{S}(t)$ and $\tilde{S}(T)$ (here, we use Theorem 5.1). Thus,

$$\tilde{\mathbb{E}}\left(\int_0^T S(t)S(T)dt\right) = \mathbb{E}\left(\int_0^T \tilde{S}(t)\tilde{S}(T)dt\right).$$

The remarkable fact is that (as in example 3 in the slides) it is possible to change the order of the two operations:

$$\mathbb{E}\left(\int_0^T \tilde{S}(t)\tilde{S}(T)dt\right) = \int_0^T \mathbb{E}\left(\tilde{S}(t)\tilde{S}(T)\right)dt.$$

In words, rather than first computing the integral and then the expectation, we can first compute the expectation and after that compute the integral.

We have that

$$\tilde{S}(t)\tilde{S}(T) = Se^{\tilde{\mu}t+\sigma W(t)} \times Se^{\tilde{\mu}T+\sigma W(T)} = S^2 e^{\tilde{\mu}(t+T)+\sigma W(t)+\sigma W(T)}$$

and hence

$$\mathbb{E}\left(\tilde{S}(t)\tilde{S}(T)\right) = S^2 e^{\tilde{\mu}(t+T)}\mathbb{E}\left(e^{\sigma(W(t)+W(T))}\right).$$

Using the result stated in (3) with $a = b = \sigma$ and $s = T - t$ we obtain

$$\mathbb{E}\left(\tilde{S}(t)\tilde{S}(T)\right) = S^2 e^{\tilde{\mu}(t+T)+2\sigma^2 t+\frac{\sigma^2}{2}(T-t)} = S^2 e^{\tilde{\mu}(t+T)+1.5\sigma^2 t+\frac{\sigma^2}{2}T}.$$

Since $\tilde{\mu} = r - \frac{\sigma^2}{2}$, we have

$$\mathbb{E}\left(\tilde{S}(t)\tilde{S}(T)\right) = S^2 e^{rT+(r+\sigma^2)t}.$$

Integrating the last expression, we obtain

$$\mathbb{E}\left(\int_0^T \tilde{S}(t)\tilde{S}(t)dt\right) = S^2 e^{rT}\int_0^T S^2 e^{(r+\sigma^2)t}dt = S^2 e^{rT}\frac{1}{r+\sigma^2}(e^{(r+\sigma^2)T} - 1).$$

Finally we obtain from (4):

$$C = \frac{S^2}{(r+\sigma^2)T}(e^{(r+\sigma^2)T} - 1).$$