MTH6112 Actuarial Financial Engineering
Coursework Week 1

You are **not supposed to submit solutions**. However, solving all the questions would help you to prepare for the exam. So, try to solve as many of them as you can. We shall discuss them during the tutorial.

1. Compute the expectation and the variance of a Brownian motion Y_t with drift parameter μ and volatility parameter σ.

   Remark. We often use the notation Y_t instead of $Y(t)$. Similar convention applies to other random processes.

   Solution

   Since $Y_t = \mu t + \sigma W_t$, we obtain $\mathbb{E}(Y_t) = \mathbb{E}(\mu t + \sigma W_t) = \mu t$. We use the fact that $\mathbb{E}(W_t) = 0$.

   Next, $\mathrm{Var}(Y_t) = \mathrm{Var}(\mu t + \sigma W_t) = \mathrm{Var}(\sigma W_t) = \sigma^2 \mathrm{Var}(W_t) = \sigma^2 t$.

   We use two facts: if X is a random variable then for any constant c one has:

   $\mathrm{Var}(X + c) = \mathrm{Var}(X)$, $\mathrm{Var}(cX) = c^2 \mathrm{Var}(X)$.

2. In lectures, we have shown that $\mathrm{Cov}(W_t, W_s) = \min(t, s)$ (here W_t is the Wiener process). Using this result (or otherwise), compute the covariance $\mathrm{Cov}(Y_t, Y_s)$ of a Brownian motion with drift parameter μ and volatility parameter σ.

   Solution Recall the following properties of the covariance.

   1. If $X,\ Y$ are random variables and $a,\ b$ are any numbers then $\mathrm{Cov}(X + a, Y + b) = \mathrm{Cov}(X, Y)$.

   2. $X,\ Y$ are random variables and $c,\ d$ are any numbers then $\mathrm{Cov}(cX, dY) = cd\mathrm{Cov}(X, Y)$.

   Simple exercise: prove these properties.

   Hence

$$\begin{aligned}\mathrm{Cov}(Y_t, Y_s) &= \mathrm{Cov}(\mu t + \sigma W_t, \mu s + \sigma W_s) \\ &= \mathrm{Cov}(\sigma W_t, \sigma W_s) \\ &= \sigma^2 \mathrm{Cov}(W_t, W_s) = \sigma^2 \min(s, t).\end{aligned}$$

3. Let $W(t)$ be a Wiener process. For a real number $a > 0$, define a new process

$$Z(t) = \sqrt{a}W(at).$$

Prove that $Z(t)$ is again a Wiener process.

Solution We have to check that the properties listed in the definition of the Wiener process are satisfied by $Z(t)$. (See the corresponding definition).

1. $Z(0) = \sqrt{a}W(a0) = \sqrt{a}W(0) = 0$ (because $W(0) = 0$).

2. We have to show that for any $t < T$ the random variable $Z(T) - Z(t) \sim \mathcal{N}(0, T - t)$.

Indeed, $Z(T) - Z(t) = \sqrt{a}(W(aT) - W(at))$ by the definition of $Z(t)$. Since $W(aT) - W(at) \sim \mathcal{N}(0, aT - at)$ (explain this!), it follows that $\sqrt{a}(W(aT) - W(at)) \sim \mathcal{N}(0, T - t)$.

Remark. We use here the following fact: if $X \sim \mathcal{N}(c, d^2)$ then $a_1 + a_2X \sim \mathcal{N}(a_1 + a_2c, a_2^2d^2)$. See the corresponding theorem in Slides Week 1.

3. Let is check that for any $0 < t_1 < t_2 < ... < t_n$ the increments $Z(t_1)$, $Z(t_2) - Z(t_1)$,... , $Z(t_n) - Z(t_{n-1})$ are independent random variables. Indeed, $Z(t_i) - Z(t_{i-1}) = \sqrt{a}(W(at_i) - W(at_{i-1}))$. Since $at_1 < at_2 < ... < at_n$, it follows that $W(at_i) - W(at_{i-1})$ are independent random variables. Hence also $Z(t_i) - Z(t_{i-1})$ are independent.

4. Finally, the continuity of $Z(t)$ follows directly from the continuity of $W(t)$. This has nothing to do with the fact that $W(t)$ is a Wiener process: if a function $f(t)$ is continuous for $t \geq 0$ then also $a_1f(a_2t)$ is continuous for any a_1 and $a_2 > 0$.

4. A share price $S(t)$ evolves according to a Geometric Brownian motion with drift parameter μ and volatility parameter σ. Compute the moments $\mathbb{E}(S(t)^m)$ for all m (which may not be an integer number). Also, compute the variance of $S(t)$.

Remark. To do that, you don't have to repeat the calculations we have carried out in the lecture.

Solution We know that if $S_t = Se^{\mu t + \sigma W_t}$ then

$$\mathbb{E}(S_t) = Se^{\mu t + \frac{\sigma^2 t}{2}}.$$

Obviously, $S_t^m = (Se^{\mu t + \sigma W_t})^m = S^m e^{m\mu t + m\sigma W_t}$. Notice that $\tilde{Y}_t = m\mu t + m\sigma W_t$ is again a Brownian motion (but with parameters $m\mu$ and $m\sigma$), and hence S_t^m is the Geometric Brownian motion with the initial value S^m and parameters $m\mu$ and $m\sigma$. Hence, by the previous result

$$\mathbb{E}(S_t^m) = S^m e^{m\mu t + \frac{m^2\sigma^2 t}{2}}.$$

It is now easy to compute the variance of S_t.

$$\begin{aligned}
\mathrm{Var}(S_t) &= \mathbb{E}(S_t^2) - (\mathbb{E}(S_t))^2 \\
&= S^2 e^{2\mu t + 2\sigma^2 t} - \left(Se^{\mu t + \frac{\sigma^2 t}{2}}\right)^2 \\
&= S^2 e^{2\mu t + 2\sigma^2 t} - S^2 e^{2\mu t + \sigma^2 t} = S^2 e^{2\mu t + \sigma^2 t}(e^{\sigma^2 t} - 1).
\end{aligned}$$