# Actuarial Financial Engineering

## Week 1

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# Reference

Ross, S., 2011 *An Elementary Introduction to Mathematical Finance*, 3rd Edition, Cambridge University Press.

Great credit and thanks to Professor Ilya Goldsheid, the previous lecturer, for his excellent work in producing the original version of these notes.

# Advice to learn this module

A lot of notations and equations are involved in this topic.
Don't panic, they are just notations and rules, and we mathematicians play the game using them.

**Draw timelines**

Timelines help you get a clear picture of all the cash flows at different time.

**Every equation tells a story**

Understand the economic meaning of each equation / math expression rather than memorize it.

# Two separate branches of Finance

There are two separate branches of Finance:

- Risk Neutral Probability - the $\mathcal{Q}$ World.
  E.g., derivative pricing.
- Real World Probability - the $\mathcal{P}$ World.
  E.g., risk and portfolio management.

# Overview of this week

## 1. Wiener Process, Brownian Motion, Geometric Brownian Motion

# 1.1. Definitions

**Definition 1.1**

A *stochastic process*, $Y(t)$, is a collection of random variables $\{Y(t)\}_{t \geq 0}$.
That is, a random variable $Y(t)$, for each $t \geq 0$.

Stochastic processes are used to model time-dependent random quantities, for example:

- the price of oil at time $t$,
- the number of users of a telephone network at time $t, \ldots$

The most basic stochastic process is *Brownian motion*.

# 1.1. Definitions

**Definition 1.2**

*Wiener process* or *Standard Brownian motion* is a stochastic process $W(t)$ such that:

1. $W(0) = 0$.
2. For any $t < T$ the random variable $W(T) - W(t)$ is **normal** with mean 0 and variance $T - t$.
   This is noted $W(T) - W(t) \sim \mathcal{N}(0, T - t)$.
3. For any $0 < t_1 < t_2 < ... < t_n$ the increments $W(t_1)$, $W(t_2) - W(t_1)$, ..., $W(t_n) - W(t_{n-1})$ are independent random variables.
4. The function $t \mapsto W_t$ is **continuous**.

# 1.1. Definitions

Remarks:

1. Property (3) above is sometimes stated in the following equivalent way:
   for any $t < T$, the randomvariable $W(T) - W(t)$ is **independent** of the trajectory $W(\tau)$, $0 \leq \tau \leq t$.
   This is the so-called the **Markov** property.
   It states that the **future** moves of Brownian motion do **not** depend on the **past**.
2. A difficult theorem shows that there actually exists a continuous stochastic process with these properties. We will assume this fact.

# 1.1. Definitions

In general Brownian motion is allowed to have different rates of increase and volatility.

**Definition 1.3**

*Brownian motion with drift* having

- **drift** parameters $\mu$ and
- **volatility** parameters $\sigma$

is the stochastic process $Y(t)$ given by:

$$Y(t) = \mu t + \sigma W(t).$$

# 1.1. Definitions

**Proposition 1.1**

If $Y(t)$ is a Brownian motion with drift, then:

1. $Y(0) = 0$.
2. For any $t < T$ the random variable $Y(T) - Y(t)$ is normal
   with mean $\mu(T - t)$ and variance $\sigma^2(T - t)$.
   This is noted $Y(T) - Y(t) \sim \mathcal{N}(\mu(T - t), \sigma^2(T - t))$.
   In particular: For any $T$ the random variable $Y(T)$ is normal with mean $\mu T$ and variance $\sigma^2 T$.
3. For any $0 < t_1 < t_2 < ... < t_n$
   the increments $Y(t_1)$, $Y(t_2) - Y(t_1)$, ..., $Y(t_n) - Y(t_{n-1})$
   are independent random variables.

## 1.1. Definitions

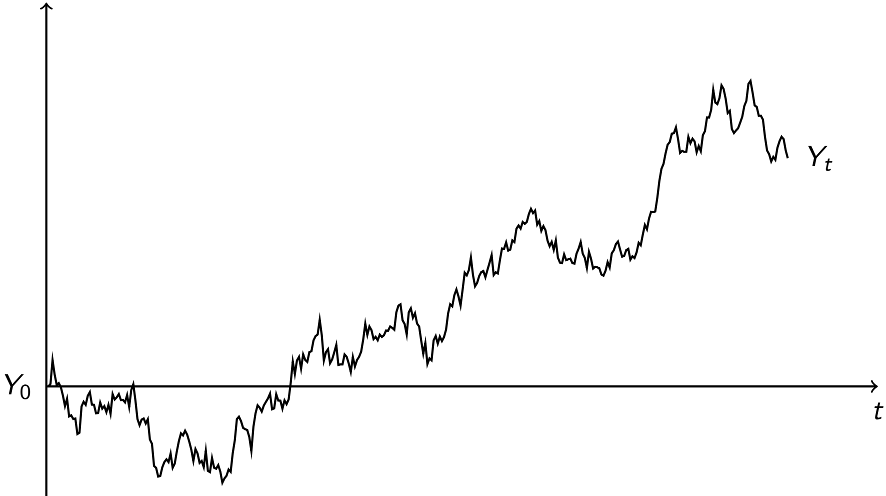
**Proof.**

We leave this as an exercise. Hints:
Write $Y$ in terms of $W$ using Definition 1.3, then:

1. Comes from the fact that $W(0) = 0$.
2. Use properties in the definition of $W$ and properties of the normal distribution.
3. Use the definition of independent variables.

□

# Sample path of a Brownian motion with drift 10% ($\mu = 0.1$) and volatility 25% ($\sigma = 0.25$).



# 1.1. Definitions

Remark:
The evolution of the **price** of a financial asset is not adequately modelled with Brownian motion because Brownian motion can easily become negative as the normal random variable $Y(t) \sim \mathcal{N}(\mu t, \sigma^2 t)$ can be negative.

Another reason why Brownian motion is not adequate is described in Ross (2011, pp.38) and is related to the fact that the size of the increment $Y(T) - Y(t)$ is independent of $Y(t)$.
This is against financial experience where the move of a stock tends to be a **percentage** move,
for example 2%, which will be bigger if the stock price is $Y(t) = \$100$ than if the stock price were $Y(t) = \$90$.

# 1.1. Definitions

A better model for the random moves of a financial asset is *Geometric Brownian Motion*.

**Definition 1.4**

A *Geometric Brownian motion* with drift $\mu$, volatility $\sigma$, and starting value $S$, is a process $S(t)$ given by

$$S(t) = S\mathrm{e}^{\mu t + \sigma W(t)},$$

where $W(t)$ is the Wiener process.

# 1.1. Definitions

Remarks:

1. Note that as $W(0) = 0$ we have that $S(0) = S$. This explains the term *starting value* in the definition above. We will often use $S(0)$ instead of $S$ to emphasize this.
2. The above definition is equivalent to stating that $S(t) = S\exp(Y(t))$ where $Y(t)$ is a Brownian motion with drift $\mu$ and volatility $\sigma$. Or to stating that $\log(S(t)) = \log(S) + Y(t) = \log(S) + \mu t + \sigma W(t)$.[1]
3. The terminology for Brownian motion and Wiener process can vary slightly in different references. You might find that what we call Wiener process in the notes is called Brownian motion by some other authors. For example Ross(2011) does not demand that Wiener process starts at 0 as we do. This does not normally lead to any problems provided one is aware of the definitions being used.

[1] Note that all logarithms in higher mathematics are in base e. In mathematics text these logarithms are written log or ln but in many calculators log refers to the decimal logarithm which we will never use. So you should be careful when using a calculator. Note for example that the derivative of $\log(x)$ is $1/x$ only if the logarithm is in base e.

# 1.2. Covariance of the Wiener process

## 1. Wiener Process, Brownian Motion, Geometric Brownian Motion

# 1.2. Covariance of the Wiener process

The Wiener process is an example of a Gaussian process.
We shall not be discussing the latter in this course, but it is useful to mention that all their properties are (in some sense) defined by their **covariance** functions.

It is useful and easy to compute the $\mathrm{Cov}(W_t, W_s)$ of the Wiener process.
Here and below we sometimes use the notation $W_t$ instead of $W(t)$.

# 1.2. Covariance of the Wiener process

**Lemma 1.1**

*If $W_t$ is the Wiener process then* $\mathrm{Cov}(W_t, W_s) = \min(t, s)$

# 1.2. Covariance of the Wiener process

**Proof.**

Recall that covariance of two random variables $X$, $Y$ is defined by

$$\mathrm{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y).$$

In our case, $\mathbb{E}(W_t) = \mathbb{E}(W_s) = 0$ and so $\mathrm{Cov}(W_t, W_s) = \mathbb{E}(W_T W_s)$.
Suppose that $t \leq s$. Then

$$\mathrm{Cov}(W_t, W_s) = \mathbb{E}(W_t W_s) = \mathbb{E}(W_t(W_s - W_t + W_t)) = \mathbb{E}(W_t(W_s - W_t)) + \mathbb{E}(W_t^2).$$

Since, by property 3 of the Wiener process, $W_t$ and $W_s - W_t$ are independent on the random variable $s$, we have that $\mathbb{E}(W_t(W_s - W_t)) = \mathbb{E}(W_t)\mathbb{E}(W_s - W_t) = 0$.
Thus $\mathrm{Cov}(W_t, W_s) = \mathbb{E}(W_t^2) = \mathrm{Var}(W_t) = t$.
If $s < t$ then a similar argument yields $\mathrm{Cov}(W_t, W_s) = \mathbb{E}(W_s^2) = s$. Thus $\mathrm{Cov}(W_t, W_s) = \min(t, s)$. □

# 1.3. Some calculations with normal random variables

## 1. Wiener Process, Brownian Motion, Geometric Brownian Motion

# 1.3. Some calculations with normal random variables

In order to better understand Geometric Brownian motion we need some basic results on normal random variables.

**Proposition 1.2**

If $X$ is the standard normal random variable with mean 0 and variance 1, $X \sim \mathcal{N}(0, 1)$, then for any $\mu \in \mathbb{R}$ and $\sigma > 0$, we have that

$$Y = \mu + \sigma X \sim \mathcal{N}(\mu, \sigma^2).$$

# 1.3. Some calculations with normal random variables

Proof:

Recall that by the definition of a normal random variable, the pdf of $X$ is $f_X(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2}}$.

We have to prove that $f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)$.

Also recall that $f_Y(y) = F'_Y(y)$, where $F_Y(y) = \mathbb{P}(Y \leq y)$ is the distribution function of $Y$. So

$$F_Y(y) = \mathbb{P}(\sigma X + \mu \leq y) = \mathbb{P}\left(X \leq \frac{y-\mu}{\sigma}\right) = F_X\left(\frac{y-\mu}{\sigma}\right)$$

(where $F_X$ is the distribution function of $X$). Thus

$$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_X\left(\frac{y-\mu}{\sigma}\right) = \frac{1}{\sigma} f_X\left(\frac{y-\mu}{\sigma}\right)$$

# 1.3. Some calculations with normal random variables

Proof (cont.):
and therefore

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$$

$\square$

Exercise:
Prove that $Y = \mu + \sigma X \sim \mathcal{N}(\mu, \sigma^2)$ also if $\sigma < 0$.
However, in this case $f_Y(y) = \frac{1}{\sqrt{2\pi}|\sigma|} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$

# 1.3. Some calculations with normal random variables

Remark:
The above result can be obtained by applying the so called **Transformation Formula** which states that if $Y = g(X)$ where $g$ is a monotone function then the pdf of $Y$ is

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{\mathrm{d}}{\mathrm{d}y} g^{-1}(y) \right| .$$

In our case $g(x) = \sigma x + \mu$, and since $g^{-1}(y) = (y - \mu)/\sigma$ one obtains the result.

# 1.3. Some calculations with normal random variables

**Corollary 1.1**

*If* $Y \sim \mathcal{N}(\mu, \sigma^2)$ *and* $Z = aY + b$*, then* $Z \sim \mathcal{N}\left(a\mu + b, a^2\sigma^2\right)$

**Proof.**

To prove this we observe that according to Proposition 1.2, $Y = \mu + \sigma X$, where $X$ is a standard normal random variable.
Hence $Z = a(\sigma X + \mu) + b = a\sigma X + (a\mu + b)$.
Once again, by Proposition 1.2, $Z \sim \mathcal{N}\left(a\mu + b, a^2\sigma^2\right)$. □

# 1.4. The expectation of the Geometric Brownian Motion

## 1. Wiener Process, Brownian Motion, Geometric Brownian Motion

# 1.4. The expectation of the Geometric Brownian Motion

In order to use GBM, $S(t)$, to model financial assets it will be important to know what the expected value, $\mathbb{E}(S(t))$ is.

Remark:
Note that the expectation of Brownian motion with drift $\mu$ and volatility $\sigma$ is easy to calculate since $Y(t) = \mu t + \sigma W(t)$ and $W(t) \sim \mathcal{N}(0, t)$, we have, by Corollary 1.1, that $Y(t) \sim \mathcal{N}(\mu t, \sigma^2 t)$. Therefore

$$\mathbb{E}\left(Y(t)\right) = \mu t \quad \text{and moreover} \quad \mathrm{Var}\left(Y(t)\right) = \sigma^2 t.$$

# 1.4. The expectation of the Geometric Brownian Motion

The calculation of the expectation of Geometric Brownian motion is more complex.
We first recall the following formula:
if $X$ is a random variable, $f_X(x)$ is its pdf, and $g : \mathbb{R} \mapsto \mathbb{R}$ is a function then

$$\mathbb{E}\left(g(X)\right) = \int_{-\infty}^{\infty} g(x) f_X(x) dx \;\text{ if }\; \int_{-\infty}^{\infty} |g(x)| f_X(x) dx < \infty. \tag{1}$$

# 1.4. The expectation of the Geometric Brownian Motion

The expectation of GBM is given by the following:

**Theorem 1.1**

*If $S(t)$ is a Geometric Brownian Motion with drift $\mu$ and volatility $\sigma$ then*

$$\mathbb{E}\left(S(t)\right) = S(0)\mathrm{e}^{\mu t + \frac{\sigma^2 t}{2}}.$$

# 1.4. The expectation of the Geometric Brownian Motion

**Proof:**

By the definition of the GBM, $S(t) = S(0)\exp(\mu t + \sigma W(t))$ and so

$$\mathbb{E}\left(S(t)\right) = S(0)\mathbb{E}\left(\mathrm{e}^{\mu t+\sigma W(t)}\right) = S(0)\mathrm{e}^{\mu t}\mathbb{E}\left(\mathrm{e}^{\sigma W(t)}\right)$$

It remains to compute $\mathbb{E}\left(\mathrm{e}^{\sigma W(t)}\right)$. $W(t) \sim \mathcal{N}(0, t)$ with $f_{W(t)}(x) = \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{x^2}{2t}\right)$ and by

$$\mathbb{E}\left(g(X)\right) = \int_{-\infty}^{\infty} g(x) f_X(x) dx \quad \text{if} \quad \int_{-\infty}^{\infty} |g(x)| f_X(x) dx < \infty. \tag{2}$$

we have:

$$\mathbb{E}\left(\mathrm{e}^{\sigma W(t)}\right) = \int_{-\infty}^{+\infty} \mathrm{e}^{\sigma x} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) \mathrm{d}x = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \exp\left(-\frac{x^2}{2t} + \sigma x\right) \mathrm{d}x.$$

# 1.4. The expectation of the Geometric Brownian Motion

**Proof (cont.):**

To compute the integral above we use a trick called completion of squares. Namely,

$$-\frac{x^2}{2t} + \sigma x = -\frac{x^2 - 2t\sigma x}{2t} = -\frac{(x - t\sigma)^2 - t^2\sigma^2}{2t} = -\frac{(x - t\sigma)^2}{2t} + \frac{t\sigma^2}{2}.$$

We thus have

$$\begin{aligned} \mathbb{E}\left(\mathrm{e}^{\sigma W(t)}\right) &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \exp\left(-\frac{(x - t\sigma)^2}{2t} + \frac{t\sigma^2}{2}\right) \mathrm{d}x \\ &= \frac{\mathrm{e}^{\frac{t\sigma^2}{2}}}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{(x-t\sigma)^2}{2t}} \mathrm{d}x = \mathrm{e}^{\frac{t\sigma^2}{2}}. \end{aligned}$$

# 1.4. The expectation of the Geometric Brownian Motion

**Proof (cont.):**

The last equality is due to the fact that $\frac{1}{\sqrt{2\pi t}}\mathrm{e}^{-\frac{(x-a)^2}{2t}}$ is the probability density function of a normal random variable $\mathcal{N}(a, t)$ and hence, for any $a$,

$$\frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{+\infty}\mathrm{e}^{-\frac{(x-a)^2}{2t}}\mathrm{d}x = 1.$$

Combining the above we get

$$\mathbb{E}\left(S(t)\right) = S(0)\mathrm{e}^{\mu t+\frac{\sigma^2 t}{2}}.\square$$

# 1.5. Computing the moments of the Wiener process

## 1. Wiener Process, Brownian Motion, Geometric Brownian Motion

# 1.5. Computing the moments of the Wiener process

The just obtained equality $\mathbb{E}\left(\mathrm{e}^{\sigma W(t)}\right) = \mathrm{e}^{\frac{\sigma^2 t}{2}}$ can be used for computing all moments of the Wiener process.
We need the following equality:

$$\mathrm{e}^{z} = 1 + z + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!} + \ldots = \sum_{n=0}^{\infty} \frac{z^n}{n!}. \tag{3}$$

# 1.5. Computing the moments of the Wiener process

**Lemma 1.2**

*For any integer $j \geq 1$*

$$\mathbb{E}(W_t^{2j}) = \frac{(2j)!}{j!2^j} t^j \text{ and } \mathbb{E}(W_t^{2j-1}) = 0.$$

# 1.5. Computing the moments of the Wiener process

**Proof:**
Equation (3) implies that

$$e^{\sigma W_t} = \sum_{n=0}^{\infty} \frac{\sigma^n W_t^n}{n!}$$

and hence

$$\mathbb{E}\left(e^{\sigma W_t}\right) = \mathbb{E}\left(\sum_{n=0}^{\infty} \frac{\sigma^n W_t^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{\sigma^n \mathbb{E}(W_t^n)}{n!}. \tag{4}$$

On the other hand, applying (3) once again, we can write

$$e^{\frac{\sigma^2 t}{2}} = \sum_{n=0}^{\infty} \frac{\sigma^{2j} t^j}{2^j j!}. \tag{5}$$

# 1.5. Computing the moments of the Wiener process

**Proof (cont.):**

Since $\mathbb{E}\left(\mathrm{e}^{\sigma W(t)}\right) = \mathrm{e}^{\frac{\sigma^2 t}{2}}$, Equation 4 = Equation 5.
We thus have that for all $\sigma$,

$$\sum_{n=0}^{\infty} \frac{\mathbb{E}(W_t^n)}{n!}\sigma^n = \sum_{n=0}^{\infty} \frac{t^j}{2^j\, j!}\sigma^{2j}.$$

In the last equality, the coefficients in front of equal powers of $\sigma$ must be the same.
Hence,
if $n = 2j-1$ then $\mathbb{E}(W_t^{2j-1}) = 0$ (because the corresponding coefficients on the right are zeros);
if $n = 2j$ then $\frac{\mathbb{E}(W_t^{2j})}{2j!} = \frac{t^j}{2^j\, j!}$ which is the same as the first equality in the statement of the Lemma.$\square$

# 1.5. Computing the moments of the Wiener process

The above calculations are formal and require justification which is beyond the scope of this course.

We shall simply use the fact that the above relations are correct.

Our next Lemma is a simple but important corollary from the above.

**Lemma 1.3**

*The* $4^{\text{th}}$ *moment of* $W_t$ *and the variance of* $W_t^2$ *are given by*

$$\mathbb{E}(W_t^4) = 3t^2, \quad \mathrm{Var}(W_t^2) = 2t^2.$$

**Proof:**

Applying the above formulae in the case when $j = 1$ and $j = 2$ we obtain,

$$\mathbb{E}(W_t^2) = \frac{(2)!}{1!2}t = t \text{ and } \mathbb{E}(W_t^4) = \frac{4!}{2!2^2}t^2 = 3t^2.$$

□