MTH5126 - Statistics for Insurance

Worksheet 2

Q1. Useful integral formula – result from lecture slides

Prove the following result.

If $f_X(x)$ is the PDF of the $N(\mu, \sigma^2)$ distribution then:

$$\int_L^U x f_X(x)dx = \mu\left[\Phi(U') - \Phi(L')\right] - \sigma\left[\phi(U') - \phi(L')\right]$$

Where

$$L' = \frac{L-\mu}{\sigma}$$

and

$$U' = \frac{U-\mu}{\sigma}$$

With $\phi(z)$ and $\Phi(z)$ being the probability density function and distribution function respectively of the standard normal distribution.

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Using the formula for $f_X(x)$ and the substitution $z = \frac{x-\mu}{\sigma}$ gives:

$$\int_L^U x\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}dx$$

$$= \int_{L'}^{U'} (\mu+\sigma z)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz$$

$$= \mu\int_{L'}^{U'} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz + \sigma\int_{L'}^{U'} z\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz$$

$$= \mu P[L' < N(0,1) < U'] + \sigma\left[-\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\right]_{L'}^{U'}$$

$$= \mu[\Phi(U') - \Phi(L')] - \sigma[\phi(U') - \phi(L')]$$

$$= \mu\left[\Phi\left(\frac{U-\mu}{\sigma}\right) - \Phi\left(\frac{L-\mu}{\sigma}\right)\right] - \sigma\left[\phi\left(\frac{U-\mu}{\sigma}\right) - \phi\left(\frac{L-\mu}{\sigma}\right)\right]$$

When $L = -\infty$ or $U = -\infty$ these formulae can be simplified using the facts that:

$$\phi(-\infty) = \phi(\infty) = 0$$

$$\Phi(-\infty) = \Phi(\infty) = 1$$

Q2.

Peace-Of-Mind Insurance plc insures a risk for which individual claim sizes (in £000s) have a distribution, X, with mean 500 and standard deviation 250. This insurer arranges excess of loss reinsurance for this risk with a retention limit of £1,000,000.

What is the proportion of claims from this risk for which the insurer expects to receive a payment

from the reinsurer if X ~ $Gamma(\alpha,\lambda)$?
Hint: You might want to find the values of the distribution parameters first.

Gamma distribution
Using the formulae for the mean and variance of a gamma distribution:

$$\frac{\alpha}{\lambda} = 500 \text{ and } \frac{\alpha}{\lambda^2} = 250^2$$

This gives:
$\alpha = 4$ and $\lambda = 0.008$
The reinsurer will make a payment if the claim size exceeds £1m. Multiplying through by 2λ and using the $2\lambda X \sim \chi^2_{2\alpha}$ relationship and the tables we get:

$$P(X > 1{,}000) = P(2\lambda X > 2{,}000\lambda) = P(\chi^2_8 > 16) = 1 - 0.9576 = 0.0424$$

Q3. R
An exponential distribution has parameter λ = 0.4. Use the in-built functions in R to:

(a) Simulate 1,000 values from this distribution, assigning this to a variable called `Exp_vector` and find the max and min of the simulated values.

(b) Plot a histogram of `Exp_vector` showing the frequencies.
Hint: Use the `hist` function.

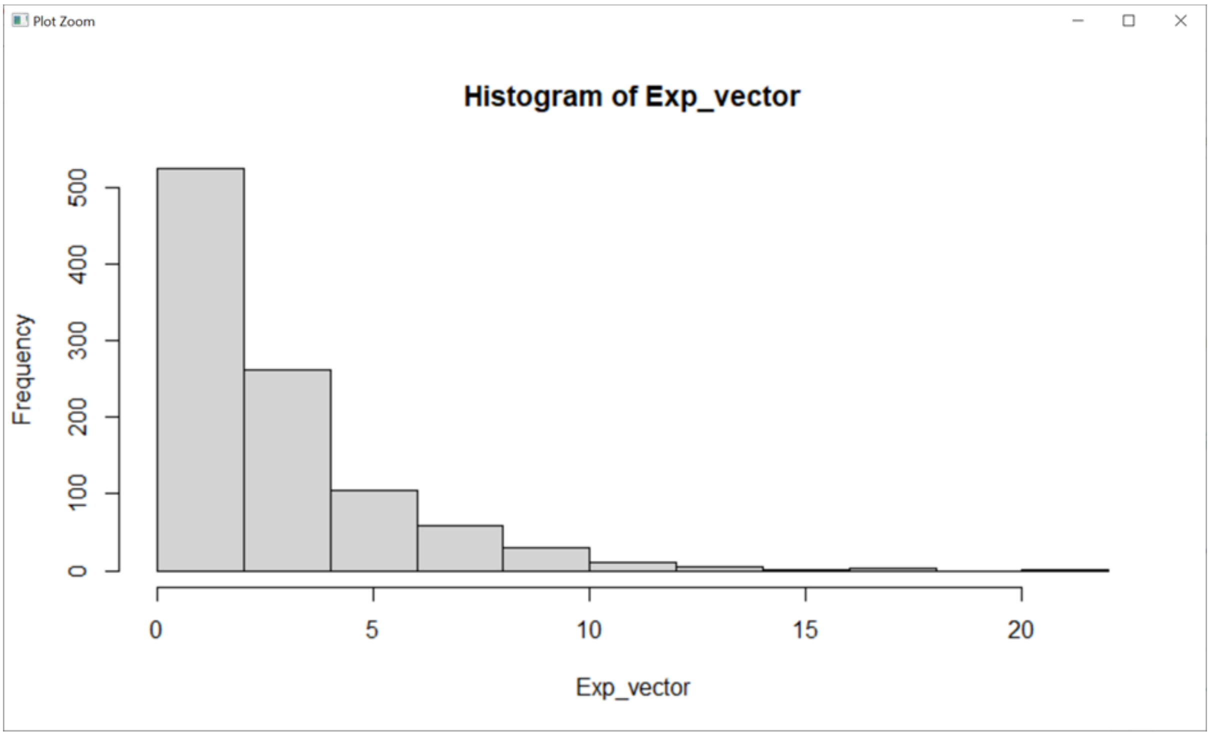
(c) Plot the theoretical density function for this distribution as:

1 a scatter plot
2 a line graph.

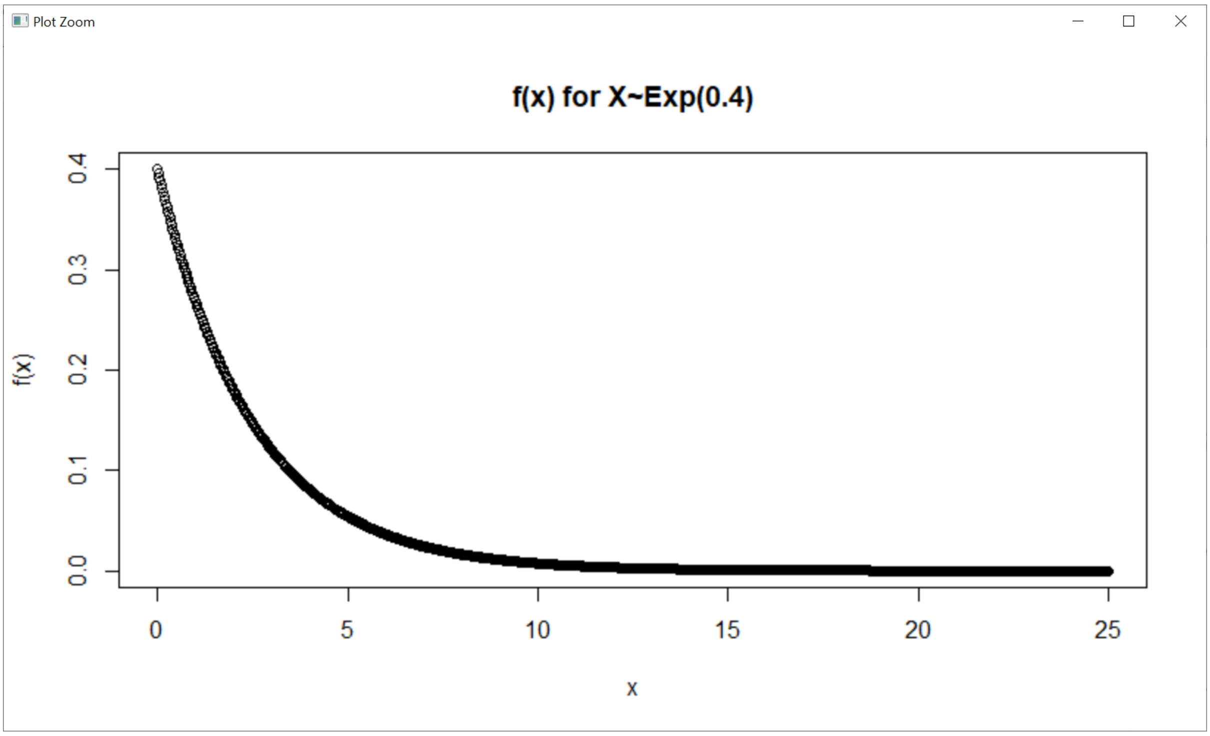
Hint: Define the range for the x axis using the `seq` function. Use the `plot` function for your plots.

```
#Q3(a) Simulate 1000 values from X~Exp(lambda)

lambda<-0.4

n<-1000

set.seed(42)

Exp_vector<-rexp(n,lambda)

> max(Exp_vector)

[1] 21.15584

> min(Exp_vector)

[1] 0.0007018867
```

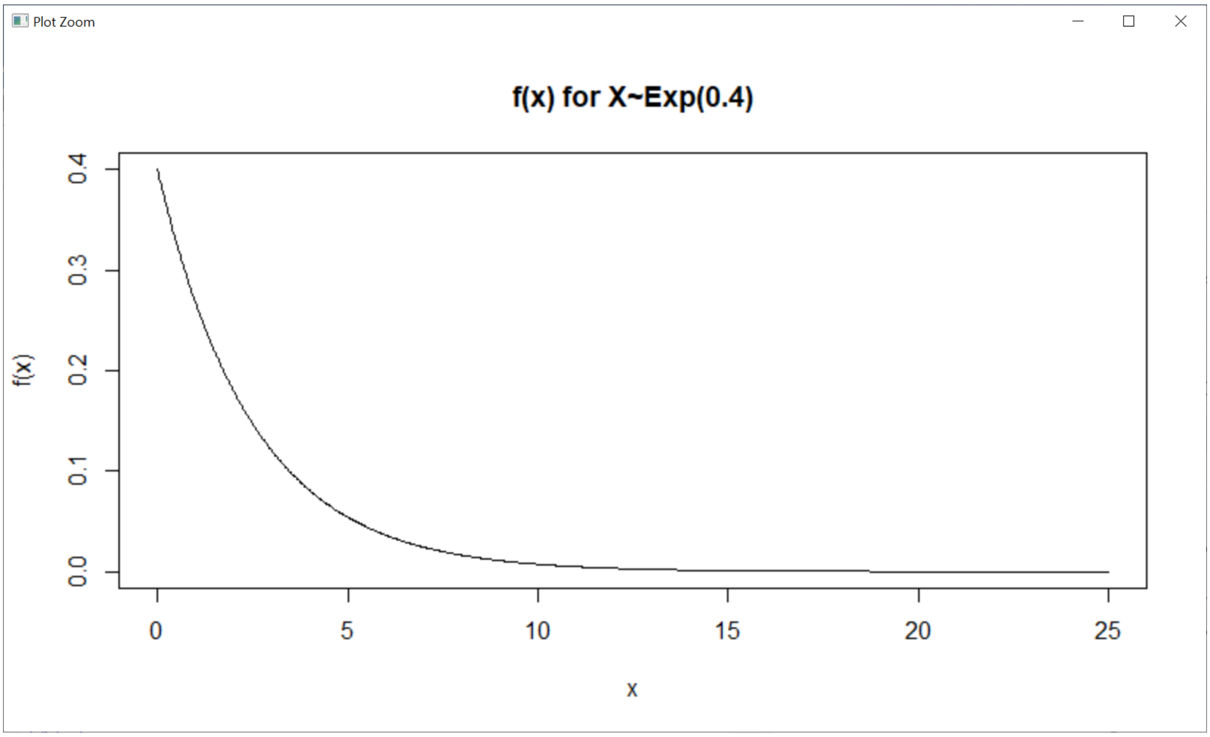
```
#Q3(b)Plot histogram
hist(Exp_vector)
```




```
#Q3(c)(1) Plot theoretical density function as a scatter plot
x<-seq(0,25,by=0.025)
fx<-dexp(x,lambda)
plot(x,fx,main="f(x) for X~Exp(0.4)",ylab="f(x)")
```


Plot Zoom
f(x) for X~Exp(0.4)
f(x)
0.0
0.1
0.2
0.3
0.4
0
5
10
15
20
25
x

```
#Q3(c)(2) Plot theoretical density function as a line graph
x<-seq(0,25,by=0.025)
fx<-dexp(x,lambda)
plot(x,fx,type="l",main="f(x) for X~Exp(0.4)",ylab="f(x)")
```



Q4. R

A lognormal distribution has parameters $\mu = 0$ and $\sigma^2 = 1$. Use the in-built functions in R to:

(a) Simulate 1,000 values from this distribution, assigning this to a vector called `LNorm_vector` and find the median of the simulated values.

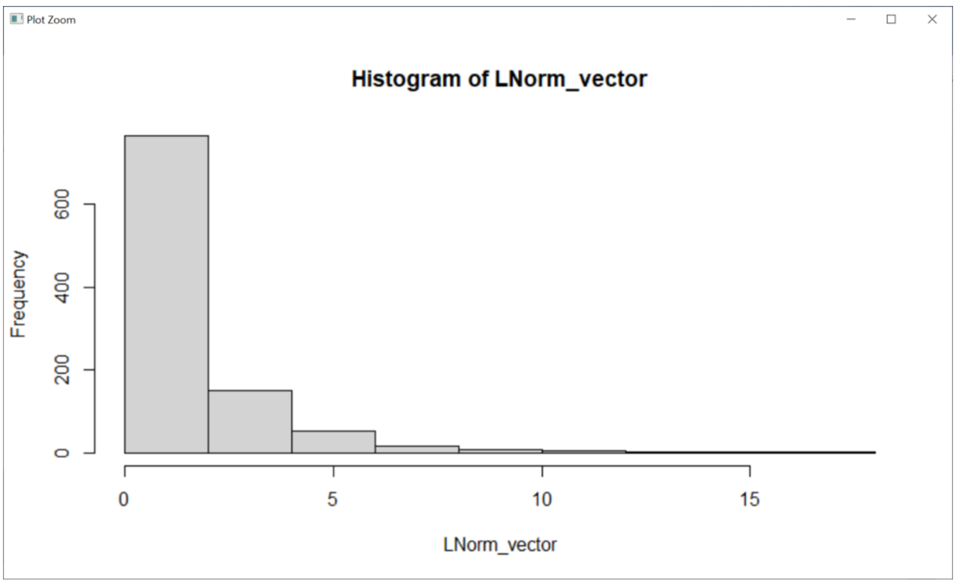
(b) Plot a histogram of `LNorm_vector` showing the frequencies.

(c) Plot a second histogram in a new graph of `LNorm_vector` showing the probability densities, setting the y-axis range from 0 to 0.7 for this graph.
Hint: In the `hist` function, set `freq=FALSE` and `ylim=c(0,0.7)`.

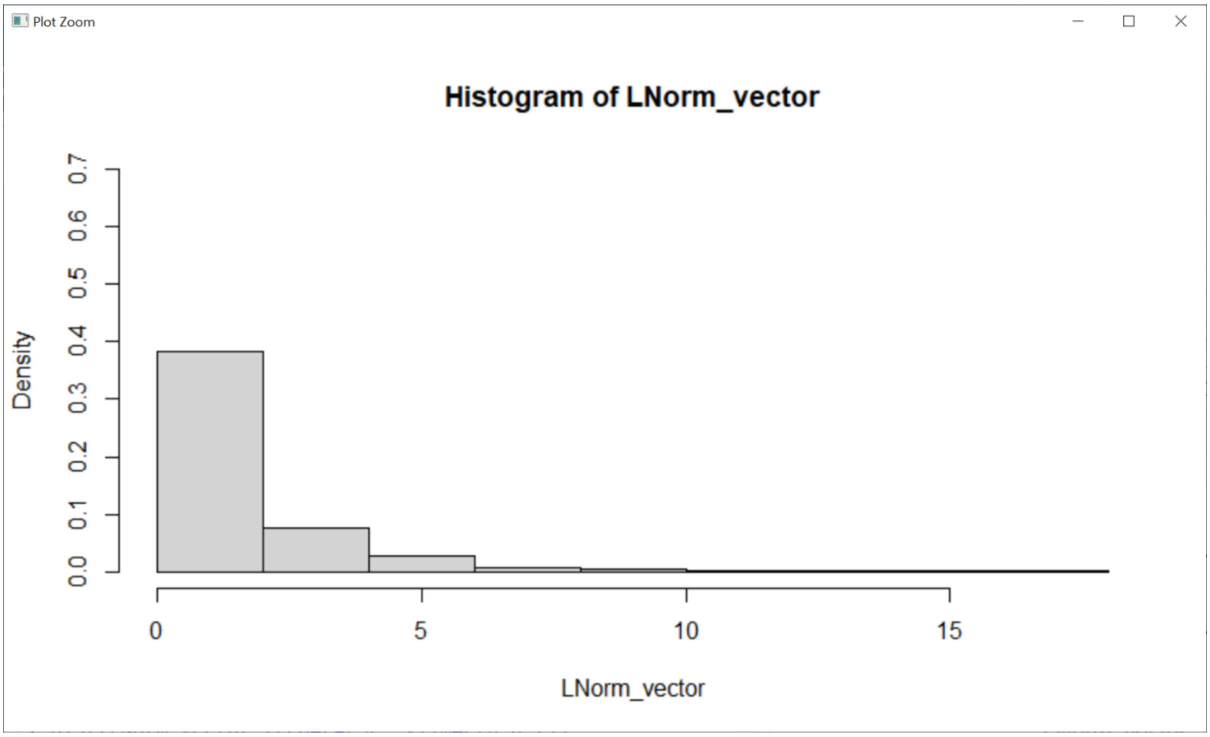
(d) Add the empirical density function of `LNorm_vector` to the chart in part (c). The empirical density function is the density function implied by the simulated values.
Hint: The `density` function computes the empirical density function. Use the function `lines` to add plots onto an existing plot.

(e) Add the theoretical density function of lognormal (0,1) distribution to the chart in part (d) to highlight the difference to the sample, including appropriate labels and legend.
Hint: Use the `legend` function to add legend to your plot.

```
#Q4(a) Simulate 1000 values from X~lognormal(mu,sigma_squared)

mu<-0
sigma<-1
set.seed(5126)
n<-1000
LNorm_vector<-rlnorm(n,mu,sigma)
> median(LNorm_vector)
[1] 1.0013

#Q4(b)Plot histogram showing frequencies
> hist(LNorm_vector)
```



```
#Q4(c) Plot new histogram showing probability densities,
setting y-axis range from 0 to 0.7
hist(LNorm_vector,freq=FALSE, ylim=c(0,0.7))
```




```
#Q4(d) Add the empirical density function of LNorm_vector
lines(density(LNorm_vector),col="red")

#Q4(e) Add the theoretical lognormal(0,1) dist
```

```
x_range<-seq(0,20,0.1)
lines(x_range,dlnorm(grid,mu,sigma),col="black")
legend(10,0.7,cex=0.3,c("Empirical density","Theoretical
density"),col=c("red","black"),lty=1)
```