MTH5126 - Statistics for Insurance

Academic Year: 2022-23 **Semester: B**

Worksheet 8

Q1. Ruin Theory

An insurer considers that claims of a certain type occur in accordance with a compound Poisson process.

The claim frequency for the whole portfolio is 100 per annum and individual claims have an exponential distribution with a mean of £8,000.

1. Calculate the adjustment coefficient if the total premium rate for the portfolio is £1,000,000 per annum.
2. Estimate the insurer's probability of ultimate ruin assuming that the initial surplus is £20,000 and future premiums remain level.

Answer:

1. R is the unique positive root to the following equation:

$$\lambda + cR = \lambda M_X(R)$$

The MGF for an exponential distribution with mean of 8,000 is

$$M_X(t) = E[e^{tX}]$$
$$= 1/(1-8{,}000t),\ t < 1/8{,}000$$

[see derivation of MGF in Lecture 1 slides 14 and 15]

So:

$$M_X(R) = 1/(1 - 8{,}000R)$$

So the adjustment coefficient satisfies:

$$100 + 1,000,000R = \frac{100}{1 - 8,000R}$$

Dividing by 100:

$$1 + 10,000R = \frac{1}{1 - 8,000R}$$

Rearranging:

$$(1 + 10,000R)(1 - 8,000R) = 1$$

$$1 + 2,000R - 80,000,000R^2 = 1$$

Cancelling the ones and factorising:

$$2,000R(1 - 40,000R) = 0$$

The adjustment coefficient is the unique positive root, i.e.

$$R = 1/\,40,000 = 0.000025$$

2. The probability of ultimate ruin, $\psi(U)$

$\approx e^{-RU}$

$= e^{-0.000025*20,000}$

$= 0.61$

Q2. Ruin Theory

Claim events on a portfolio of insurance policies follow a Poisson process with parameter λ. Individual claim amounts follow a distribution X with density

$$f(x) = 0.01^2 x e^{-0.01x}, \qquad x > 0.$$

The insurance company calculates premiums using a premium loading of 45%.

1. Derive the moment generating function $M_X(t)$.
2. Determine the adjustment coefficient.
3. Find the surplus required to ensure the probability of ultimate ruin is less than 1%.

Answer:

1.

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} f(x)dx$$
$$= \int_0^\infty 0.01^2 x e^{(t-0.01)x} dx$$

We use integration by parts to solve the above:
Let

$$u = 0.01^2 x$$
$$\frac{dv}{dx} = e^{(t-0.01)x}$$

Then

$$\frac{du}{dx} = 0.01^2$$
$$v = \int e^{(t-0.01)x}\, dx = \frac{e^{(t-0.01)x}}{(t-0.01)}$$

So

$$M_X(t) = [uv]_0^\infty - \int_0^\infty v \frac{du}{dx}\, dx$$

$$= \left[\frac{0.01^2 x e^{(t-0.01)x}}{t-0.01}\right]_0^\infty - \int_0^\infty \frac{0.01^2 e^{(t-0.01)x}}{t-0.01} dx$$
$$= 0 - 0 - \left[\frac{0.01^2 e^{(t-0.01)x}}{(t-0.01)^2}\right]_0^\infty$$

(provided that $t < 0.01$).

$$= \frac{0.01^2}{(t-0.01)^2}$$

2. The adjustment coefficient is the unique positive solution of

$$\lambda + cR = \lambda M_X(R)$$

Rearranging,

$M_X(R) = 1 + \frac{cR}{\lambda}$

$= 1 + (1+\theta)\, E(X)\, R$, using $c = (1+\theta)\, \lambda\, E(X)$

$= 1 + 1.45\, E(X)\, R$

But:

$$E(X) = M_X'(0) = \frac{d}{dt}\left[\frac{0.01^2}{(t-0.01)^2}\right]_{t=0}$$

$$= \left.\frac{-2\times 0.01^2}{(t-0.01)^3}\right|_{t=0} = \frac{-2}{-0.01} = 200$$

So we need to solve

$$\frac{0.01^2}{(R-0.01)^2} = 1 + 290R$$

$$\Longrightarrow\ 0.01^2 = (1+290R)(R-0.01)^2 = (1+290R)(0.01^2 - 0.02R + R^2)$$

$$\Longrightarrow\ 0.01^2 = 0.01^2 + 0.029R - 0.02R - 5.8R^2 + R^2 + 290R^3$$

$$\Longrightarrow\ 290R^2 - 4.8R + 0.009 = 0$$

$$R = \frac{4.8 \pm \sqrt{4.8^2 - 4\times 290\times 0.009}}{2\times 290}$$

i.e., $R = 0.00215578$ or $R = 0.0143959$

The $M_X(t)$ in this question is defined for $t < 0.01$, so we take the solution which is < 0.01, i.e.

$$R = 0.00215578.$$

3. The upper bound for the probability of ultimate ruin is given by Lundberg's inequality as:

$$\psi(U) \le e^{-RU}$$

We want $\psi(U) \le e^{-RU} = e^{-0.00215578U} < 0.01$

$\Rightarrow\ -0.00215578U < \ln 0.01$

$\Rightarrow\ U > \ln 0.01/(-0.00215578) = 2136.20$

Further practice:

As usual, after each lecture and seminar, check that you can now do the lecture examples/questions and seminar questions without looking at the answers.