MTH5126 - Statistics for Insurance

Academic Year: 2022-23 **Semester: B**

Worksheet 7 - Solutions

Q1. Copula

(i) Write down Sklar's theorem.

(ii) Explain, in words, the meaning of the following copula expression: $C(u, v, w)$

The Gumbel copula has a generating function:

$$\psi(F(x)) = (-\ln F(x))^{\alpha} \text{ where } 1 \leq \alpha < \infty$$

(iii) Derive an expression for the Gumbel (Hougaard) copula for the case where there are three variables.

A student has fitted a Gumbel copula to investment returns from three developing markets, and has calculated a value for the dependency parameter, α, of 4.0.

She has separately determined that the probability of making a loss over the next calendar year (i.e., the probability that the return is less than 0%) in each of the three markets is 5%, 7.5% and 10% respectively.

(iv) Calculate the probability that all three markets have returns of less than 0% over the next calendar year.

(v) State what type of copula is equivalent to a Gumbel copula if $\alpha = 1.0$.

(vi) Calculate the probability that all three markets have returns of less than 0% over the next calendar year, assuming that each of the markets were independent.

Answer:

(i) Let F be a joint distribution function with marginal cumulative distribution functions $F_1,\dots,F_d$. Then there exists a copula C such that for all $x_1,\dots,x_d \in [-\infty, \infty]$:

$$F(x_1,\dots,x_d) = C[F_1(x_1),\dots,F_d(x_d)]$$

(ii) $C[u, v, w]$

This gives the probability that RV1 is in the bottom u percentile, and RV2 is in the bottom v percentile, and RV3 is in the bottom w percentile.

(iii) The Gumbel generating function is defined:

$$\Psi(t) = (-\ln t)^{\alpha}$$

• Check $\Psi(0) = \lim_{t\to 0}(-\ln t)^{\alpha} = \infty$, so the pseudo-inverse equals the normal inverse

• Now invert the relationship $x = (-\ln y)^{\alpha}$ to find the normal inverse function:

$$y = \exp(-x^{1/\alpha})$$

• So, $C(u, v, w) = \Psi^{[-1]}(\Psi(u) + \Psi(v) + \Psi(w))$
$= \Psi^{(-1)}(\Psi(u) + \Psi(v) + \Psi(w))$, since the pseudo-inverse equals the normal inverse
$= \Psi^{(-1)}([-\ln u]^{\alpha} + [-\ln v]^{\alpha} + [-\ln w]^{\alpha})$
$= \exp(- \{ [-\ln u]^{\alpha} + [-\ln v]^{\alpha} + [-\ln w]^{\alpha} \}^{1/\alpha})$

(iv) P(all three markets have returns of less than 0% over the next calendar year)
$= C(0.05, 0.075, 0.1)$
$= \exp(- \{ [-\ln 0.05]^{4} + [-\ln 0.075]^{4} + [-\ln 0.1]^{4} \}^{0.25})$
= 2.957%

(v) Independent OR Independence OR Product

Check: If $\alpha = 1$,
$C(u, v, w) = \exp(- \{ [-\ln u]^{1} + [-\ln v]^{1} + [-\ln w]^{1} \}^{1/1})$
$= \exp(\ln u + \ln v + \ln w)$
$= \exp(\ln uvw)$
$= uvw$

(vi)
EITHER:
P(all three markets have returns of less than 0% over the next calendar year)
= 0.05 * 0.075 * 0.1 = 0.0375%

OR:
P(all three markets have returns of less than 0% over the next calendar year)
=exp (-((-ln 0.05)^1 + (-ln 0.075)^1 + (-ln 0.1)^1)^1)
= exp (ln 0.05+ ln 0.075 + ln 0.1)
= 0.05 * 0.075 * 0.1 = 0.0375%

Q2. Copula

Let X and Y be two random variables representing the future lifetimes of two 40-year old individuals. The two lives are married. You are given that:

$P(X \leq 20) = 0.17831$ and $P(Y \leq 20) = 0.11086$

Calculate the joint probability that both lives will die by the age of 60 using the Frank copula with $\alpha = 5$.

Answer:

$P(X \leq 20,\ Y \leq 20)$

$= C\,[u,\ v\,]$, where $u = P(X \leq 20)$, $v = P(Y \leq 20)$

$$= -\frac{1}{\alpha}\ln\left(1 + \frac{\left(e^{-\alpha u} - 1\right)\left(e^{-\alpha v} - 1\right)}{\left(e^{-\alpha} - 1\right)}\right)$$

$$= -\frac{1}{5}\ln\left(1 + \frac{\left(e^{-5\times 0.17831} - 1\right)\left(e^{-5\times 0.11086} - 1\right)}{\left(e^{-5} - 1\right)}\right)$$

$= 0.0583$

Q3. **Copula**

(i) Derive the coefficient of upper tail dependence for the Gumbel copula.
(ii) Comment on how the value of the parameter α affects the degree of upper tail dependence in the case of the Gumbel copula.

Answer:
(i) The coefficient of upper tail dependence

$$\lambda_U = \lim_{u \to 1^-} \left(\frac{1 - 2u + C[u,u]}{1-u} \right)$$

For the Gumbel copula setting $u = v$ gives:

$$C[u,u] = \exp\left\{ -\left((-\ln u)^{\alpha} + (-\ln u)^{\alpha} \right)^{1/\alpha} \right\}$$

$$= \exp\left\{ -\left(2(-\ln u)^{\alpha} \right)^{1/\alpha} \right\}$$

$$= \exp\left\{ -\left(2^{1/\alpha} (-\ln u) \right) \right\}$$

$$= \exp\left\{ \left(2^{1/\alpha} \ln u \right) \right\}$$

$$= \exp\left\{ \ln u^{2^{1/\alpha}} \right\}$$

$$= u^{2^{1/\alpha}}$$

The coefficient of upper tail dependence is given by:

$$\lambda_U = \lim_{u \to 1^-} \frac{1 - 2u + C[u,u]}{1-u} = \lim_{u \to 1^-} \frac{1 - 2u + u^{2^{1/\alpha}}}{1-u}$$

In the limit this fraction has the form $\frac{0}{0}$, which is undefined. However, we can use L'Hôpital's rule, $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$, to find the value of the limit:

$$\lambda_U = \lim_{u \to 1^-} \frac{1 - 2u + u^{2^{1/\alpha}}}{1-u} = \lim_{u \to 1^-} \frac{-2 + 2^{1/\alpha} u^{2^{1/\alpha}-1}}{-1} = 2 - 2^{1/\alpha}$$

(ii) As α increases, $2^{1/\alpha}$ reduces and hence $2 - 2^{1/\alpha}$ increases. So, increasing the value of the parameter α increases the degree of upper tail dependence of the Gumbel copula.