Coursework #1 - Question 3

Find the general solution of

$$(x+3y-2)y' = e^{-3(x+3y-2)^2} - y - \frac{1}{3}(x-2)$$

Solution : Rearranging we get

$$(x+3y-2)\,y' = e^{-3(x+3y-2)^2} - \frac{1}{3}(x+3y-2)$$

Putting $z = x+3y-2$ we get

$$z\,y' = e^{-3z^2} - \frac{1}{3}z \implies y' = \frac{e^{-3z^2}}{z} - \frac{1}{3} = f(z)$$

The ODE is of the type $y' = f(ax+by+c) = f(z)$

Reducible to separable.

Using $z = x+3y-2$ and differentiating

$$z' = 1 + 3y' \overset{\text{using } y' = f(z)}{=} 1 + 3f(z)$$

$$\Rightarrow \quad z' = 1 + 3\left(\frac{e^{-3z^2}}{z} - \frac{1}{3}\right) = 3\frac{e^{-3z^2}}{z}$$

This ODE for $z = z(x)$ is separable

Integrating we get

$$\frac{dz}{dx} = 3\frac{e^{-3z^2}}{z} \Rightarrow \int z e^{3z^2} dz = 3\int dx + C$$

$$H(z) = \int z e^{3z^2} dz = \int \frac{1}{6} d\left(e^{3z^2}\right) = \frac{1}{6} e^{3z^2}$$

$$F(x) = 3\int dx = 3x$$

$$\frac{1}{6} e^{3z^2} = 3x + C$$

$$e^{3z^2} = 18x + 6C \Rightarrow 3z^2 = \ln|18x + C'|$$

$$z = \pm\sqrt{\frac{1}{3}\ln(18x + c')}$$

Using $z = x + 3y - 2 \Rightarrow y = \frac{z - x + 2}{3}$

$$y = \frac{1}{3}\left[\pm\sqrt{\frac{1}{3}\ln(18x + c')} - x + 2\right]$$

$\square$

where c' is an arbitrary constant