MTH5104 CONVERGENCE AND CONTINUITY (2023-2024)
QUIZ 4 (WEEK 10)

Upload your solutions as a single PDF as an answer to the QMPlus quiz. Make sure that your scans are legible before uploading them.

Important: All your answers must be justified. Unless the question explicitly indicates otherwise, you may use any result from the lecture notes, provided you state the result clearly.

Determine whether the following series converge. You do not need to calculate their values.

(1) $\displaystyle\sum_{k=1}^{\infty} \frac{1+3^k}{1+2^k}$.

(5 marks)

(2) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{2^k+k}$.

(5 marks)

Solutions.

(1) This series does not converge. We will prove this using the ratio test. Given a series $\sum_{k=1}^{\infty} x_k$ with $x_k > 0$ for all $k \in \mathbb{N}$, the ratio test says that if $x_{k+1}/x_k \to x$ with $x > 1$, then the series does not converge. In our example, we have

$$\begin{aligned}\frac{x_{k+1}}{x_k} &= \frac{1+3^{k+1}}{1+2^{k+1}} \cdot \frac{1+2^k}{1+3^k} \\ &= \frac{1+3^{k+1}}{1+3^k} \cdot \frac{1+2^k}{1+2^{k+1}} \\ &= \frac{(\frac{1}{3})^{k+1}+1}{(\frac{1}{3})^{k+1}+\frac{1}{3}} \cdot \frac{(\frac{1}{2})^{k+1}+\frac{1}{2}}{(\frac{1}{2})^{k+1}+1} \\ &\to \frac{1}{1/3} \cdot \frac{1/2}{1} = \frac{3}{2} > 1.\end{aligned}$$

Applying the ratio test, we conclude that the series does not converge.

(2) A result from lectures states that if a series is absolutely convergent, then it is convergent. It therefore suffices to show that the series is absolutely convergent. The associated series of absolute values is

$$\sum_{k=1}^{\infty} \frac{1}{2^k+k}$$

and we have:

$$\frac{1}{2^k+k} \leq \frac{1}{2^k}$$

for all $k \in \mathbb{N}$. The series

$$\sum_{k=1}^{\infty} \frac{1}{2^k}$$

converges by a result from lectures, which states that a geometric series with common ratio r converges if $|r| < 1$. In this case the common ratio is $1/2$, and certainly $|1/2| < 1$.

Finally the comparison test states that if (x_k) and (y_k) are sequences, with $0 \leq x_k \leq y_k$ for all $k \in \mathbb{N}$, and if $\Sigma_{k=1}^{\infty} y_k$ exists, then $\Sigma_{k=1}^{\infty} x_k$ exists. We apply this to our example, taking

$$x_k = \frac{1}{2^k + k}, \qquad y_k = \frac{1}{2^k}.$$

Since $\Sigma_{k=1}^{\infty} y_k$ converges, we conclude that $\Sigma_{k=1}^{\infty} x_k$ converges. The original series is thus absolutely convergent, and so *a fortiori* convergent.