Queen Mary
University of London

MTH5104: Convergence and Continuity 2023–2024
Problem Sheet 6 (Continuity)

1. Prove, directly for the definition of continuity, that $f(x) = \sqrt[3]{x}$ is continuous at $a = 0$.

Solution. At $a = 0$,

$$|f(x) - f(a)| = |\sqrt[3]{x} - \sqrt[3]{0}\,| = \sqrt[3]{|x|} = \sqrt[3]{|x-a|}.$$

Given $\varepsilon > 0$, let $\delta = \varepsilon^3$. Then, for any x with $|x-a| < \delta$, we have $|f(x)-f(a)| = \sqrt[3]{|x-a|} < \sqrt[3]{\delta} = \varepsilon$. Thus, f is continuous at $a = 0$.

2. For each of the following functions, state whether they are continuous at $a = 0$ and *prove* your answers, using only the definition of continuity.

(a) $f(x) = \begin{cases} 2x & \text{if } x \in \mathbb{Q}, \\ -5x & \text{if } x \notin \mathbb{Q}, \end{cases}$

(b) $f(x) = \begin{cases} 2x+1 & \text{if } x \geq 0, \\ -5x & \text{if } x < 0. \end{cases}$

Solution. This is a question from the May 2015 Exam.

(a) $f(x) = \begin{cases} 2x & \text{if } x \in \mathbb{Q}, \\ -5x & \text{if } x \notin \mathbb{Q}, \end{cases}$ is continuous at $a = 0$. We have to prove

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall h \in \mathbb{R}, |h| < \delta\ :\ |f(a+h) - f(a)| < \varepsilon.$$

Given $\varepsilon > 0$, pick $\delta = \dfrac{\varepsilon}{5} > 0$. Then for any h with $|h| < \delta$, we have

$$|f(a+h) - f(a)| = |f(h) - f(0)| = |f(h)| \leq 5|h| < 5\delta = \varepsilon.$$

This proves the claim.

(b) $f(x) = \begin{cases} 2x+1 & \text{if } x \geq 0, \\ -5x & \text{if } x < 0, \end{cases}$ is *not* continuous at $a = 0$. We thus have to prove the negation of the above quantifier statement, i.e.

$$\exists \varepsilon > 0\ \forall \delta > 0\ \exists h \in \mathbb{R}, |h| < \delta\ :\ |f(a+h) - f(a)| \geq \varepsilon.$$

Pick $\varepsilon = \frac{1}{2}$. Then given $\delta > 0$, pick $h = -\min\{\frac{1}{10}, \frac{\delta}{2}\}$. This is allowed as $|h| < \delta$. Moreover, as $h \in [-\frac{1}{10}, 0)$, we have $f(h) = -5h \leq \frac{1}{2}$. On the other hand $f(0) = 1$, so $|f(h) - f(0)| \geq \frac{1}{2} = \varepsilon$.

3. The function $f : \mathbb{R} \to \mathbb{R}$ is defined by:

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Z}, \\ x & \text{if } x \in \mathbb{Z}. \end{cases}$$

Find the set $P \subseteq \mathbb{R}$ of points for which the function f is not continuous. Prove your answer!

Solution. We claim that the set P of points where f is not continuous is $\mathbb{Z} \setminus \{0\}$.

Before starting the proof, let us first recall that f is continuous at a iff

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall h \in \mathbb{R}, |h| < \delta\ :\ |f(a+h) - f(a)| < \varepsilon. \tag{0.1}$$

f is not continuous at a iff the negation of (1) holds, i.e.

$$\exists \varepsilon > 0\ \forall \delta > 0\ \exists h \in \mathbb{R}, |h| < \delta\ :\ |f(a+h) - f(a)| \geq \varepsilon. \tag{0.2}$$

(a) We first show that f is indeed not continuous at $a \in P = \mathbb{Z} \setminus \{0\}$. So let such an a be given. We have to prove (2). So we pick $\varepsilon = 1$. Then given $\delta > 0$, choose any h with $0 < |h| < \delta$. We get

$$|f(a+h) - f(a)| \geq 1 = \varepsilon.$$

(b) Next, we show that f is continuous on the complement of P, i.e. we want to show (1). We first prove this for $a = 0$. Given $\varepsilon > 0$, pick $\delta = 1$. Then given any h with $|h| < \delta$ we have $f(h) = 0 = f(0)$, so

$$|f(h) - f(0)| = 0 < \varepsilon.$$

Next, we let $a \in \mathbb{R} \setminus \mathbb{Z}$ and show (1) again for such an a. We pick $\delta = \min\{|a - k|\ :\ k \in \mathbb{Z}\}$. Now given any h with $|h| < \delta$, we have $f(a+h) = 0$ (as $(a - \delta, a + \delta)$ does not contain any elements of $\mathbb{Z}$) and $f(a) = 0$ (as $a \notin \mathbb{Z}$). Therefore

$$|f(a+h) - f(a)| = 0 < \varepsilon.$$

4. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 + x$. Prove, directly from the definition, that $f(x)$ is continuous at all $a \in \mathbb{R}$. (There is an example in the notes that can be used as a model. Given ε, you may like to try letting $\delta = \min\{c\varepsilon, 1\}$ for some suitably chosen constant $c \in \mathbb{R}$.)

(b) Let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} x^2 + x, & \text{if } x \text{ is rational} \\ 0, & \text{otherwise.} \end{cases}$$

The function g is continuous at two points; what are they?

(c) For one of the points a you identified in part (b), verify that g is continuous at a. (This part requires very little calculation. There are two cases, $a + h \in \mathbb{Q}$ and $a + h \notin \mathbb{Q}$. In part (a) you already did the work for the harder one of these!)

Solution.

(a) Let $a \in \mathbb{R}$. Given $\varepsilon > 0$, let $\delta = \min\{c\varepsilon, 1\}$, following the hint. Then, for any h with $|h| < \delta$,

$$\begin{aligned} |f(a+h) - f(a)| &= |((a+h)^2 + (a+h)) - (a^2 + a)| \\ &= |2ah + h^2 + h| \\ &\le |2ah| + |h^2| + |h| \\ &< |2a|\delta + \delta^2 + \delta && \text{(since } |h| < \delta) \\ &\le |2a|\delta + \delta + \delta && \text{(since } \delta \le 1) \\ &\le (|2a| + 2)\delta \\ &\le (|2a| + 2)c\varepsilon. \end{aligned}$$

Now choose $c = (|2a| + 2)^{-1}$ and we are done.

(b) The only points where g is continuous are the roots of f, namely -1 and 0. [To see why g is not continuous elsewhere, see Example 5.4(ii).]

(c) Let's check that g is continuous at 0. Given ε choose δ as in part (a). Suppose $|h| < \delta$. If h is rational then $|g(0+h)-g(0)| = |f(0+h)+f(0)| < \varepsilon$ as we saw in part (a). If h is irrational, then $|g(0 + h) - g(0)| = |0 - 0| = 0 < \varepsilon$.

Remark. If we didn't have the hint to guide us, we could do a preliminary calculation to estimate $|f(x) - f(a)|$ at $x = a + h$:

$$\begin{aligned} |f(a+h) - f(a)| &= |((a+h)^2 + (a+h)) - (a^2 + a)| \\ &= |2ah + h^2 + h| \\ &\le |2ah| + |h^2| + |h| \end{aligned}$$

We need to choose δ so that this quantity to less than the $\varepsilon > 0$ given by the Demon. We make our task easier by first insisting that δ (and hence $|h|$) is less than 1. Our goal is now to make

$$|2ah| + |h^2| + |h| \le |2a|\delta + \delta + \delta = (|2a| + 2)\delta$$

less than or equal to ε. This we can do by choosing $\delta = (|2a| + 2)^{-1}$.

5. Prove parts (i) and (ii) of Theorem 5.14 from the lecture notes.

Solution. Recall that Theorem 5.14 states that if $f : D_1 \to \mathbb{R}$ and $g : D_2 \to \mathbb{R}$ are two functions that are continuous at $a \in D_1 \cap D_2$, and $c \in \mathbb{R}$ is a constant, then the following hold:

(i) cf is continuous at a.
(ii) $f + g$ is continuous at a.
(iii) $f \cdot g$ is continuous at a.
(iv) If $g(a) \neq 0$, then $\frac{f}{g}$ is continuous at a.

We prove (i) and (ii) here and leave (iii) and (iv) to question 6.

(i) If $c = 0$, then cf is the constant function 0 which is continuous everywhere. If $c \neq 0$, then given ε, we can find a $\delta > 0$ such that $|f(x) - f(a)| < \tilde{\varepsilon} := \frac{\varepsilon}{|c|}$ for all x with $|x - a| < \delta$. (We can do this since f is continuous at a.) Now, $|cf(x) - cf(a)| < \varepsilon$ for all x with $|x - a| < \delta$, so cf is continuous at a as required.

(ii) We must prove that for $D = D_1 \cap D_2$

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in D, |x - a| < \delta \ : \ |(f(x) + g(x)) - (f(a) + g(a))| < \varepsilon.$$

But given $\varepsilon > 0$, we know that

$$\exists \delta_1 > 0 \ \forall x \in D_1, |x - a| < \delta_1 \ : \ |f(x) - f(a)| < \tilde{\varepsilon} := \frac{\varepsilon}{2}$$

because f is continuous at a, and

$$\exists \delta_2 > 0 \ \forall x \in D_2, |x - a| < \delta_2 \ : \ |g(x) - g(a)| < \tilde{\varepsilon} := \frac{\varepsilon}{2}$$

because g is continuous at a. So taking $\delta = \min\{\delta_1, \delta_2\}$, we obtain

$$\forall x \in D, |x - a| < \delta \ : \ |f(x) - f(a)| < \frac{\varepsilon}{2}$$

(from $D \subseteq D_1$ and $\delta \leq \delta_1$) as well as

$$\forall x \in D, |x - a| < \delta \ : \ |g(x) - g(a)| < \frac{\varepsilon}{2}$$

(from $D \subseteq D_2$ and $\delta \leq \delta_2$). Hence, for all $x \in D$ with $|x - a| < \delta$, we have

$$|(f(x) + g(x)) - (f(a) + g(a))| \leq |f(x) - f(a)| + |g(x) - g(a)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

by the triangle inequality. Thus $f + g$ is continuous at a.

6. For each of the following real functions state its natural domain of definition D. Also determine at which points of D the function is continuous. Explain your answers by reference to results in the course. You may assume that $\ln x$ is defined and continuous at all points in $(0, \infty)$.

   (a) $f(x) = \cos\left(\frac{1}{x^2+1}\right)$,

   (b) $g(x) = \ln(\ln x)$, and

   (c) $h(x) = \sqrt{\frac{x}{x^2+1}}$.

   Solution. (Below, "continuous" will mean "defined and continuous".)

   (a) The function f is continuous on the whole of $\mathbb{R}$.
   The numerator and denominator of the quotient are polynomials and hence are continuous on the whole of $\mathbb{R}$. The denominator always strictly positive, so $1/(x^2+1)$ is continuous on $\mathbb{R}$ by Theorem 5.14(iv). Cosine is continuous on $\mathbb{R}$, and so is $f(x)$ by Theorem 5.17.

   (b) The function g is continuous on the whole of $(1, \infty)$.
   The outer logarithm will be defined iff $\ln(x) > 0$. This in turn will occur iff $x > 1$. So the function g is defined on $(1, \infty)$ (and not elsewhere). By Theorem 5.17, the function g is also continuous on $(1, \infty)$, since $\ln(x)$ is continuous at all points in its domain.

   (c) The function is continuous on $[0, \infty)$.
   As in (a), the quotient is continuous on $\mathbb{R}$. However, when $x < 0$ the quotient is negative, and the square root is not defined. So the natural domain of the function is $[0, \infty)$. Square root is continuous on $[0, \infty)$ (result from the module), and so is $h(x)$ by Theorem 5.17.

7. Consider the function $f(x) = \sqrt{x}$ defined on $D = [0, \infty)$. Prove, directly from the definition of continuity, that $f(x)$ is continuous on D. (First, try showing that $|\sqrt{x} - \sqrt{a}\,|^2 \leq |x - a|$.)

   Solution. Following the hint, note that

$$|\sqrt{x} - \sqrt{a}\,|^2 \leq |\sqrt{x} - \sqrt{a}\,| \times |\sqrt{x} + \sqrt{a}\,| = |(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})| = |x - a|.$$

   This suggests that to achieve $|\sqrt{x} - \sqrt{a}\,| < \varepsilon$ it is enough to have $|x - a| < \varepsilon^2$. So given $\varepsilon > 0$, set $\delta = \varepsilon^2$. Then for all $x \in [0, \infty)$ with $|x - a| < \delta$ we have $|f(x) - f(a)| = |\sqrt{x} - \sqrt{a}\,| \leq \sqrt{|x - a|}$ (by the hint) and hence $|f(x) - f(a)\,| < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon$.

8. **Challenge.** Prove parts (iii) and (iv) of Theorem 5.14 from the lecture notes.

Solution. We continue from Question 2.

(iii) This part is more tricky. Similar to the prove of the corresponding result for convergent sequences, the idea is to write

$$\begin{aligned} f(x)g(x) - f(a)g(a) &= (f(x) - f(a))(g(x) - g(a)) \\ &\quad + f(a)(g(x) - g(a)) + g(a)(f(x) - f(a)). \end{aligned}$$

and then choose δ such that the modulus of each of the 3 terms is $< \frac{\varepsilon}{3}$ for all x with $|x - a| < \delta$. So if you couldn't solve this, try it again with this idea in mind! What follows is a detailed argument.

We have to show that the following statement is true:

$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in D_1 \cap D_2, |x - a| < \delta\ :\ |f(x) \cdot g(x) - f(a) \cdot g(a)| < \varepsilon.$

So given any $\varepsilon > 0$, to ensure that the modulus of the left hand side of this equation is less than ε it will suffice to ensure that:

- $|(f(x) - f(a))(g(x) - g(a))| < \varepsilon/3$,
- $|f(a)(g(x) - g(a))| < \varepsilon/3$, and
- $|g(a)(f(x) - f(a))| < \varepsilon/3$.

Since f and g are continuous at a we can choose δ_1 such that for all $x \in D_1$ with $|x - a| < \delta_1$ we have $|f(x) - f(a)| < \sqrt{\varepsilon/3}$ and we can choose δ_2 such that for all $x \in D_2$ with $|x - a| < \delta_2$ we have $|g(x) - g(a)| < \sqrt{\varepsilon/3}$.

Moreover, if $f(a) \neq 0$ we can choose δ_3 such that if $x \in D_2$ with $|x - a| < \delta_3$ then $|g(x) - g(a)| < \dfrac{\varepsilon}{3|f(a)|}$ – and if $f(a) = 0$ we can choose any value of δ_3, for example $\delta_3 = 1$.

Similarly, if $g(a) \neq 0$ we can choose δ_4 such that if $x \in D_1$ with $|x - a| < \delta_4$ then $|f(x) - f(a)| < \dfrac{\varepsilon}{3|g(a)|}$ – and if $g(a) = 0$ we can choose any value of δ_4, for example $\delta_4 = 1$.

Thus, finally, if we set $\delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4\}$ then for every $x \in D_1 \cap D_2$ such that $|x - a| < \delta$ we have

$$\begin{aligned} |(f(x) - f(a))(g(x) - g(a))| + |f(a)(g(x) - g(a))| + |g(a)(f(x) - f(a))| \\ < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

So the claim follows by the triangle inequality.

(iv) This statement follows similarly to iii). One can also use that $h(x) = \frac{1}{x}$ is continuous away from zero and thus for continuous $g(x)$ we have $\frac{1}{g(x)} = h(g(x))$ is continuous as long as $g(x) \neq 0$. Then using iii), we conclude that $\frac{f}{g}$ is continuous. Details can be found in every textbook on the topic.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \text{ or if } x = 0, \\ 1/q & \text{if } x \in \mathbb{Q} \text{ and } x = p/q \text{ in lowest terms, with } p > 0. \end{cases}$$

(a) Suppose that $a \in \mathbb{Q}$. Prove that f is not continuous at a.

(b) Suppose that $a \notin \mathbb{Q}$. Prove that f is continuous at a [harder].

Solution. Let $f : \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \text{ or if } x = 0, \\ 1/q & \text{if } x \in \mathbb{Q} \text{ and } x = \frac{p}{q} \text{ in lowest terms, with } p > 0, \end{cases}$$

(a) Let $a \in \mathbb{Q}$. We claim that f is not continuous at a.

Proof. We pick $\varepsilon = |f(a)| = |1/q|$. Then given $\delta > 0$ we pick an *irrational* $x \in (a - \delta, a + \delta)$ (which we can do as irrational numbers are dense by Corollary 1.15). Then

$$|f(x) - f(a)| = |0 - f(a)| = |f(a)| \geq \varepsilon.$$

□

(b) Let $a \notin \mathbb{Q}$. We claim that f is continuous at a. [This is harder than anything that would be asked in the exam, so a solution is only briefly outlined here.]

Proof. Given $\varepsilon > 0$, choose a natural number q' such that $1/q' < \varepsilon$. For example, for definiteness we could take $q' = \lceil 1/\varepsilon \rceil$. There are only *finitely* many rational numbers of the form p/q which have $|a - p/q| < 1$ and also have $q < q'$. Let z be the closest such rational number to a.

Pick $\delta = |z - a|$ and note that $\delta > 0$ as $z \neq a$, since a is irrational and z is rational. Then given any x with $|x - a| < \delta$, we have two possibilities:

i. If x is irrational then

$$|f(x) - f(a)| = |0 - 0| = 0 < \varepsilon.$$

ii. If x is rational, say $x = p/q$ then we know q must be greater or equal to q' since we chose z to be the closest rational to a which has denominator less than q, and $|x - a| < \delta = |z - a|$. Hence

$$|f(x) - f(a)| = |1/q - 0| = 1/q \leq 1/q' < \varepsilon.$$

In either case $|f(x) - f(a)| < \varepsilon$ so f is continuous at a. □

10. Using the intermediate value theorem, show that the following equations have a solution $x \in \mathbb{R}$:

    (a) $x^5 + 2x^2 = 1$.

    (b) $x^4 + 1 = 9x$.

    (c) $x\cos(x) + x^2 = 1$.

    In all cases, move all terms onto the left-hand side to get an equation of the form $f(x) = 0$. Then prove that the function $f(x)$ is continuous. Finally, find $a, b \in \mathbb{R}$ such that $f(a) < 0$ and $f(b) > 0$. The Intermediate Value Theorem then says that there exists a real number c in between a and b such that $f(c) = 0$.

11. Find a continuous map of the open interval $(0, 1) \subset \mathbb{R}$ to itself which has no fixed point in $(0, 1)$. This shows that the analogue of the Brouwer Fixed Point Theorem for *open* intervals is not true.

    Solution. We aim to find a continuous map of the open interval $(0, 1) \subset \mathbb{R}$ to itself which has no fixed point. We claim that $f(x) = (x + 1)/2$ is such a map.

    Proof. f is obviously continuous (since it is the sum of $x \to x/2$ and $x \to 1/2$, which are continuous), and f maps the open interval $(0, 1)$ to the open interval $(1/2, 1) \subset (0, 1)$.

    Moreover, since the unique solution in $\mathbb{R}$ to the equation $(x + 1)/2 = x$ is $x = 1$, the map f has no fixed point in the open interval $(0, 1)$. □

12. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function which has the property that $f(0) = f(1)$. Let $g : [0, 1/2] \to \mathbb{R}$ be defined by $g(x) = f(x + 1/2) - f(x)$. Show that $g(0) + g(1/2) = 0$. By applying the Intermediate Value Theorem to g, prove that there exists a real number $c \in [0, 1/2]$ such that $f(c+1/2) = f(c)$.

    Solution. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function which has the property that $f(0) = f(1)$. Let $g : [0, 1/2] \to \mathbb{R}$ be defined by $g(x) = f(x+1/2) - f(x)$. We claim that $g(0) + g(1/2) = 0$.

    Proof. From the definition of g we have $g(0) + g(1/2) = f(1/2) - f(0) + f(1) - f(1/2) = f(1) - f(0) = 0$. □

    Then we claim that there exists a real number $c \in [0, 1/2]$ such that $f(c+1/2) = f(c)$.

Proof. First observe that g is continuous (since it is a sum of continuous functions). As $g(0) + g(1/2) = 0$, there are three possibilities:

(a) $g(0) = 0$. In this case $f(0 + 1/2) - f(0) = 0$ so $f(0 + 1/2) = f(0)$.

(b) $g(0) < 0$ and $g(1/2) > 0$. Then by the Intermediate Value Theorem there exists $c \in [0, 1/2]$ such that $g(c) = 0$ and hence $f(c + 1/2) = f(c)$.

(c) $g(0) > 0$ and $g(1/2) < 0$. Then again by the IVT there exists $c \in [0, 1/2]$ such that $g(c) = 0$ and hence $f(c + 1/2) = f(c)$. □

COMMENT. As $f(0) = f(1)$ the function f induces a continuous map from the unit circle S^1 (the interval $[0, 1]$ with its ends glued together) to $\mathbb{R}$. So this exercise implies that given any continuous map from the unit circle to $\mathbb{R}$ there exists a pair of opposite points on the circle which map to the same point of $\mathbb{R}$. This is the simplest case of the Borsuk-Ulam Theorem, which is true in every dimension. The 2-dimensional case tells us that given any continuous map from the 2-sphere (e.g. the earth's surface) to the plane $\mathbb{R}^2$, there is always at least one pair of opposite points on S^2 which map to the same point. Thus, for example, at every moment there is always at least one pair of opposite points on the earth's surface which have the same temperature and the same atmospheric pressure.

13. Suppose that $f : [0, 1] \to \mathbb{R}$ and that $(x_n)_{n=1}^{\infty}$ is the sequence of all rationals ordered by increasing denominator

$$0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \dots$$

and that $f(x_n) = n$. Why do we know, *without doing any calculation* that f is not continuous? (This question does not ask for a precise proof, just an explanation.)

Solution. Suppose that $f : [0, 1] \to \mathbb{R}$ and that $(x_n)_{n=1}^{\infty}$ is the sequence of rationals

$$0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \dots$$

and that $f(x_n) = n$. We claim that f is not continuous.

Proof. Suppose it were continuous (we aim for a contradiction). Then, by the Bolzano-Weierstrass Theorem, the sequence $(x_n)_{n=1}^{\infty}$ has a converging subsequence $(x_{r_n})_{n=1}^{\infty}$ with $x_{r_n} \to x \in [0, 1]$. Now, since f is continuous at x, we obtain that $f(x_{r_n}) \to f(x) \in \mathbb{R}$, but we have that $f(x_{r_n}) = r_n$ tends to infinity. This is the desired contradiction. Actually this shows that f cannot be continuous at any point, since every point $x \in [0, 1]$ is the limit of some subsequence of $(x_n)_{n=1}^{\infty}$. □

Remark. Another way of solving this question is by using the boundedness principle, which says that a continuous function on a closed bounded interval (such as $[0, 1]$) has be bounded. Obviously the function f is not bounded above and hence cannot be continuous.

14. In this question, be sure to check that all the conditions of the Intermediate Value Theorem hold.

   (a) Prove that, for every $c \in [0, \infty)$, the equation $xe^x = c$ has a solution.

   (b) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function that is bounded, i.e, $|f(x)| \leq M$ for all $x \in \mathbb{R}$. Prove that the function $f(x)$ has a fixed point in $\mathbb{R}$.

Solution.

   (a) Consider the function $g(x) = xe^x - c$. Note that xe^x is continuous on $\mathbb{R}$ as it is the product of two continuous functions. Also, $g(x)$ is continuous, being the difference of two continuous functions.

   Now, $g(0) = 0 - c \leq 0$ and $g(c) = ce^c - c = c(e^c - 1) \geq 0$. So, by the IVT, $g(x) = 0$ for some $x \in [0, c]$. This x satisfies $xe^x = c$.

   (b) Let $g(x) = f(x) - x$. Since $f(x)$ and the identity function are both continuous, we know that g is continuous. Note that $g(-M) = f(-M) + M \geq 0$ and $g(M) = f(M) - M \leq 0$. So, by the IVT, $g(x) = 0$ for some $x \in [-M, M]$. It follows that f has a fixed point in the same interval.

15. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is given by

$$f(x) = \begin{cases} x - 1 & \text{if } x \in \mathbb{Q}, \\ x + 1 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

   (a) What does the sequence $(x_n)_{n=1}^{\infty}$ given by $x_n = \frac{n-\sqrt{2}}{n}$ converge to?

   (b) Does the sequence $(f(x_n))_{n=1}^{\infty}$ converge? If so what does it converge to?

   (c) Is f continuous at the point $a = 1$? (Give a brief justification.)

Solution.

   (a) Obviously $\lim_{n\to\infty} x_n = 1$, by Theorem 3.24, since $x_n = 1 - \sqrt{2}/n$.

   (b) First, every x_n is irrational since $\sqrt{2}$ is irrational. Hence $f(x_n) = x_n + 1 = 2 - \sqrt{2}/n$ (since x_n is irrational we are in the second case in the definition of f). This converges to 2.

(c) We prove f is not continuous by contradiction. Suppose f is continuous then by Theorem 5.19 $f(x_n) \to f(1)$ since $x_n \to 1$. But $f(x_n) \to 2 \neq f(1) = 0$. Hence f is not continuous.

16. **Challenge**. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Recall from Definition 5.22 that we say that $\lim_{x\to a} f(x)$ exists and is equal to ℓ iff

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall h \in \mathbb{R}, 0 < |h| < \delta\ :\ |f(a+h) - \ell| < \varepsilon. \qquad (\star)$$

(a) Show that $\lim_{x\to a} f(x) = \ell$ (according to the above definition) if and only if for *every* sequence $(x_n)_{n=1}^{\infty}$ which satisfies $x_n \neq a$ for all n as well as $x_n \to a$ for $n \to \infty$, we get $f(x_n) \to \ell$ as $n \to \infty$.

(b) Show that f is continuous at a (according to our Definition 5.1) if and only if $\lim_{x\to a} f(x)$ exists and is equal to $f(a)$.

Solution.

(a) We first prove that if $\lim_{x\to a} = \ell$ (i.e. statement $(\star)$ holds), then every sequence (x_n) with $x_n \neq a$ and $x_n \to a$ satisfies $f(x_n) \to \ell$, i.e.

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n > N\ :\ |f(x_n) - \ell| < \varepsilon. \qquad (\star\star)$$

So given $\varepsilon > 0$, by $(\star)$ we know that there exists $\delta > 0$ such that

$$\forall h \in \mathbb{R}, 0 < |h| < \delta\ :\ |f(a+h) - \ell| < \varepsilon.$$

But as $x_n \to a$, we also know that for this $\delta > 0$, there exists $N \in \mathbb{N}$ such that $\forall n > N\ :\ |x_n - a| < \delta$. We pick exactly this N. Then for all $n > N$, we have $0 < |x_n - a| < \delta$ and thus (with $h = x_n - a$)

$$|f(a+h) - \ell| = |f(x_n) - \ell| < \varepsilon.$$

Next, we want to prove the other direction, namely that if $(\star\star)$ holds for all sequences (x_n) with $x_n \neq a$ and $x_n \to a$, then also $(\star)$ must be true. We prove this by contradiction. We assume (towards a contradiction) that $(\star)$ is false, i.e. there exists an $\varepsilon > 0$ such that

$$\forall \delta > 0\ \exists h \in \mathbb{R}, 0 < |h| < \delta\ :\ |f(a+h) - \ell| \geq \varepsilon.$$

In particular, for this $\varepsilon > 0$ and $\delta_n = \frac{1}{n}$, we know that there exists $h_n \in \mathbb{R}$ with $0 < |h_n| < \frac{1}{n}$ such that $|f(a+h_n) - \ell| \geq \varepsilon$. We set $x_n = a + h_n$. As $|h_n| > 0$, we see that $x_n \neq a$ for all $n \in \mathbb{N}$ and as $|h_n| < \frac{1}{n}$ (and therefore $h_n \to 0$), we see that $x_n \to a$. So (x_n) is a sequence which must satisfy $(\star\star)$. But on the other hand by construction it satisfies $|f(x_n) - \ell| \geq \varepsilon$ for all n, which is the desired contradiction.

(b) If f is continuous then for *any sequence* $x_n \to a$, we know by Theorem 5.19 that $f(x_n) \to f(a)$. Thus ($\star\star$) is true (with $\ell = f(a)$ and hence by part (a), we conclude that $\lim_{x\to a} f(x)$ converges and equals $\ell = f(a)$.

Conversely, if $\lim_{x\to a} f(x)$ converges and equals $\ell = f(a)$, then ($\star$) yields

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall h \in \mathbb{R}, 0 < |h| < \delta :\ |f(a+h) - f(a)| < \varepsilon.$$

As the statement obviously also holds for $h = 0$, we find

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall h \in \mathbb{R}, |h| < \delta :\ |f(a+h) - f(a)| < \varepsilon,$$

which is the definition of f being continuous at a.