# Coursework 3 Solutions:

## PS7 Q7: (10 points)

Method 1. By the expression for general solutions of Laplace equations in polar coordinates, $u$ must be of the form

$$u(r,\theta) = C_0 + D_0 \ln r + \sum_{m=1}^{\infty} \left(C_m \cdot r^m + \frac{D_m}{r^m}\right)(A_m \cos m\theta + B_m \sin m\theta)$$

and thus $u(\frac{1}{r},\theta)$ must be of the form

(by replace $r$ with $\frac{1}{r}$ above)

$$u(\tfrac{1}{r},\theta) = C_0 + D_0 \ln(r^{-1}) + \sum_{m=1}^{\infty} \left(\frac{C_m}{r^m} + D_m r^m\right)(A_m \cos m\theta + B_m \sin m\theta)$$

so if we chose

$\widehat{C_0} = C_0$, $\widehat{D_0} = -D_0$, $\widehat{C_m} = D_m$, $\widehat{D_m} = C_m$, $\widehat{A_m} = A_m$, $\widehat{B_m} = B_m$,

we will have

$$u(\tfrac{1}{r},\theta) = \widehat{C_0} + \widehat{D_0} \ln r + \sum_{m=1}^{\infty} \left(\widehat{C_m} r^m + \frac{\widehat{D_m}}{r^m}\right)(\widehat{A_m} \cos m\theta + \widehat{B_m} \sin m\theta)$$

is also of the form of a solution to Laplace equation,

so $u(\frac{1}{r},\theta)$ is also harmonic.

Method 2.

$$\Delta\left[u(\tfrac{1}{r},\theta)\right]$$

$$= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left[u(\tfrac{1}{r},\theta)\right]$$

$$= \frac{\partial^2}{\partial r^2}\left[u(\tfrac{1}{r},\theta)\right] + \frac{1}{r}\frac{\partial}{\partial r}\left[u(\tfrac{1}{r},\theta)\right] + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\left[u(\tfrac{1}{r},\theta)\right]$$

$$= \frac{\partial}{\partial r}\left[u_r(\tfrac{1}{r},\theta)\cdot\frac{-1}{r^2}\right] + \frac{1}{r}u_r(\tfrac{1}{r},\theta)\frac{-1}{r^2} + \frac{1}{r^2}u_{\theta\theta}(\tfrac{1}{r},\theta)$$

$$= u_{rr}(\tfrac{1}{r},\theta)\cdot\left(\frac{-1}{r^2}\right)^2 + \frac{2}{r^3}u_r(\tfrac{1}{r},\theta) - \frac{1}{r^3}u_r(\tfrac{1}{r},\theta) + \frac{1}{r^2}u_{\theta\theta}(\tfrac{1}{r},\theta)$$

$$= \frac{1}{r^4}u_{rr}(\tfrac{1}{r},\theta) + \frac{1}{r^3}u_r(\tfrac{1}{r},\theta) + \frac{1}{r^2}u_{\theta\theta}(\tfrac{1}{r},\theta)$$

$$= \frac{1}{r^4}\left[ u_{rr}(\tfrac{1}{r},\theta) + r u_r(\tfrac{1}{r},\theta) + r^2 u_{\theta\theta}(\tfrac{1}{r},\theta)\right]$$

$$= \frac{1}{r^4}\left[ u_{rr}(\tfrac{1}{r},\theta) + \frac{1}{\frac{1}{r}} u_r(\tfrac{1}{r},\theta) + \frac{1}{(\frac{1}{r})^2} u_{\theta\theta}(\tfrac{1}{r},\theta)\right]$$

choosing $(r,\theta) = (\frac{1}{r_0}, \theta_0)$, we get.

$$\Delta\left[u(\tfrac{1}{r},\theta)\right]$$

$$= r_0^4 \left[ u_{rr}(r_0,\theta_0) + \frac{1}{r_0} u_r(r_0,\theta_0) + \frac{1}{r_0^2} u_{\theta\theta}(r_0,\theta_0)\right]$$

$$= r_0^4 \cdot 0$$

$$= 0$$

The bracket is zero because $u(r,\theta)$ is harmonic at $(r_0,\theta_0)$

Since $(r_0,\theta_0)$ can be chosen arbitrary, we see

$u(\frac{1}{r},\theta)$ is also harmonic !

PS 8 Q1(2): (10 points)

In cartesian coordinate $\theta = \arctan \frac{y}{x}$

To see it's harmonic, we check

$$\Delta\theta = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\theta$$

$$= 0 + 0 + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}(\theta)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial\theta}(1)$$

$$= 0$$

PS 8 Q3: (20 points)

The general solution is

$$u(r,\theta) = C_0 + D_0 \ln r + \sum_{m=1}^{\infty}\left(C_m r^m + \frac{D_m}{r^m}\right)(A_m \cos m\theta + B_m \sin m\theta)$$

plug in $r=\frac{1}{2}$ and $r=2$, using the boundary conditions, we get

$$17+17\cos 2\theta+17\sin 2\theta = u(\tfrac{1}{2},\theta)$$
$$= C_0+D_0\ln(\tfrac{1}{2})+\sum_{m=1}^{\infty}\left(\frac{C_m}{2^m}+D_m\cdot 2^m\right)(A_m\cos m\theta+B_m\sin m\theta)$$

$$17+17\cos 2\theta+17\sin 2\theta = u(2,\theta)$$
$$= C_0+D_0\ln 2+\sum_{m=1}^{\infty}\left(C_m\cdot 2^m+\frac{D_m}{2^m}\right)(A_m\cos m\theta+B_m\sin m\theta)$$

By the orthogonality of $\cos m\theta$ and $\sin n\theta$'s, we observe that $A_m, B_m=0$ for all $m\neq 2$, so

$$17+17\cos 2\theta+17\sin 2\theta = C_0-D_0\ln 2+\left(\frac{C_2}{2^2}+2^2D_2\right)(A_2\cos 2\theta+B_2\sin 2\theta)$$
$$17+17\cos 2\theta+17\sin 2\theta = C_0+D_0\ln 2+\left(C_2\cdot 2^2+\frac{D_2}{2^2}\right)(A_2\cos 2\theta+B_2\sin 2\theta)$$

we can choose $A_2=B_2=1$ and get

$$17=C_0-D_0\ln 2$$
$$17=C_0+D_0\cdot\ln 2$$
$$17=\frac{C_2}{4}+4D_2$$
$$17=4C_2+\frac{D_2}{4}$$

Solve it, we get $C_0=17$, $D_0=0$, $C_2=4$, $D_2=4$

Thus

$$u(r,\theta)=17+\left(4r^2+\frac{4}{r^2}\right)(\cos 2\theta+\sin 2\theta)$$

PS9 Q2 : (10 points)

$$\left(\frac{\partial}{\partial t}-k\frac{\partial^2}{\partial x^2}\right)\left(e^{ax+bt}\right)$$
$$= b\cdot e^{ax+bt}-k\cdot a^2e^{ax+bt}$$

$$= (b - \kappa a^2) e^{ax+bt}$$

so any $(a, b)$ satisfying $b = \kappa a^2$

will make $e^{ax+bt}$ a solution to heat equation.

PS9 Q3 (20 points)

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By change of variables, suppose first $u(x,t) = X(x)T(t)$

then $X\dot{T} = \kappa X''T$

$$\frac{\dot{T}}{\kappa T} = \frac{X''}{X} = -\lambda$$

we get 2 ODEs

$$\begin{cases} X'' = -\lambda X & ① \\ \dot{T} = -\lambda \kappa T & ② \end{cases}$$

Combining ① with the boundary values give the eigenvalue problem

$$\begin{cases} X'' = -\lambda X \\ X(0) = 0, \quad X'(\frac{\pi}{2}) = 0 \end{cases}$$

claim: $\lambda \geq 0$.

proof of claim: Multiply both sides by $X$ and integrate,

we get $\int_0^{\frac{\pi}{2}} X \cdot X'' = \int_0^{\frac{\pi}{2}} -\lambda X^2$

using integration by parts, we get

$$X \cdot X' \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (X')^2 = -\lambda \int_0^{\frac{\pi}{2}} X^2$$

The boundary conditions imply $(X \cdot X')\Big|_0^{\frac{\pi}{2}} = 0$.

So $\int_0^{\frac{\pi}{2}} (X')^2 = \lambda \int_0^{\frac{\pi}{2}} X^2$

and thus $\lambda \geq 0$.

Thus the eigenvalue problem has solution.

$X(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$

$X'(x) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$

The first boundary value gives

$0 = c_1 \cdot \cos 0 + c_2 \cdot 0 = c_1 + 0$

so $c_1 = 0$ and $c_2 \neq 0$

The second boundary condition gives

$0 = c_2 \cdot \sqrt{\lambda} \cdot \cos(\sqrt{\lambda} \cdot \frac{\pi}{2})$

so $\sqrt{\lambda} \cdot \frac{\pi}{2} = n\pi - \frac{\pi}{2}$

The eigenvalues are $\lambda_n = (2n-1)^2$, $n = 1, 2, \cdots$

The eigenfunctions are $X_n(x) = \sin[(2n-1)x]$

Knowing $\lambda_n$, the solution for $T$ is $T_n(t) = e^{-(2n-1)^2 kt}$

Thus the general solutions are

$u(x,t) = \sum_{n=1}^{\infty} a_n X_n(x) T_n(t)$

$= \sum_{n=1}^{\infty} a_n e^{-(2n-1)^2 kt} \sin[(2n-1)x]$

The initial value gives

$\sin x = u(x,0) = \sum_{n=1}^{\infty} a_n \sin[(2n-1)x]$.

We "observe" that $a_n = 0$ for all $n \neq 1$

and $a_1 = 1$ .

so $u(x,t) = e^{-k t} \sin x$

PS 10 Q3 : (15 points)

By Fourier-Poisson formula

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k t}} e^{-\frac{|x-y|^2}{4kt}} f(y)\,dy$$

with $f(x) = \begin{cases} 3, & x<0 \\ 1, & x>0 \end{cases}$ ,

so

$$u(x,t) = \int_0^{\infty} \frac{1}{\sqrt{4\pi k t}} e^{-\frac{|x-y|^2}{4kt}} dy + \int_{-\infty}^{0} \frac{3}{\sqrt{4\pi k t}} e^{-\frac{|x-y|^2}{4kt}} dy$$

using change of variables

$s = \frac{y-x}{\sqrt{4kt}}$ we have $\sqrt{4kt}\,ds = dy$

$$u(x,t) = \int_{-\frac{x}{\sqrt{4kt}}}^{\infty} \frac{1}{\sqrt{4\pi k t}} e^{-s^2} \sqrt{4kt}\,ds + \int_{-\infty}^{-\frac{x}{\sqrt{4kt}}} \frac{3}{\sqrt{4\pi k t}} e^{-s^2} \sqrt{4kt}\,ds$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k t}} e^{-s^2} \sqrt{4kt}\,ds + \int_{-\frac{x}{\sqrt{4kt}}}^{\infty} \frac{2}{\sqrt{4\pi k t}} e^{-s^2} \sqrt{4kt}\,ds$$

$$= \frac{1}{\sqrt{\pi}} \cdot \int_{-\infty}^{\infty} e^{-s^2} ds + \frac{2}{\sqrt{\pi}} \int_{-\frac{x}{\sqrt{4kt}}}^{\infty} e^{-s^2} ds$$

$$= \frac{\sqrt{\pi}}{\sqrt{\pi}} + \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-s^2} ds + \frac{2}{\sqrt{\pi}} \int_{-\frac{x}{\sqrt{4kt}}}^{0} e^{-s^2} ds$$

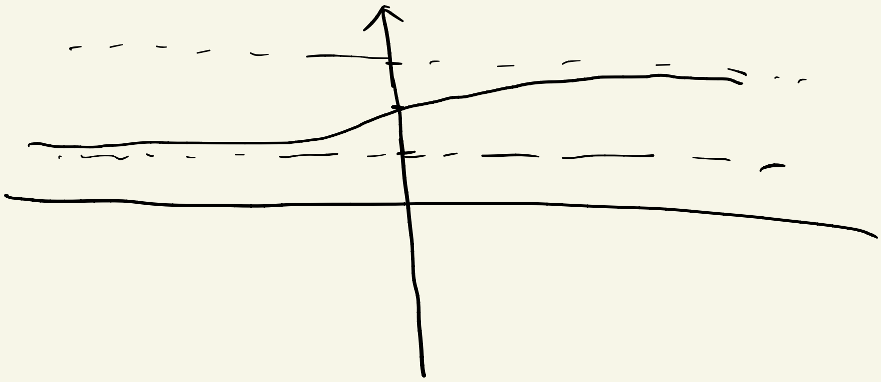
$$\left(\begin{array}{c}\text{using} \\ e^{-s^2} \text{ is even}\end{array}\right) = 1 + \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} + \frac{2}{\sqrt{\pi}} \int_{-\frac{x}{\sqrt{4kt}}}^{0} e^{-s^2} ds$$

$e^{-s^2}$ is an even function, so

$$\int_{-\frac{x}{\sqrt{4kt}}}^{0} e^{-s^2} ds = \int_{0}^{\frac{x}{\sqrt{4kt}}} ds$$

$$= 1 + 2 \cdot \frac{1}{2} + 2 \cdot \int_{0}^{\frac{x}{\sqrt{4kt}}} e^{-s^2} ds$$

$$= 1 + 2 \cdot Q(x,t)$$



As $t \to \infty$, we have

$$\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} 1 + 2\left[\frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_{0}^{\frac{x}{\sqrt{4kt}}} e^{-s^2} ds\right]$$

$$= \lim_{t\to\infty} 1 + 2\left[\frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^0 e^{-s^2} ds\right]$$

$$= 1 + 2 \cdot \frac{1}{2}$$

$$= 2$$

PS 10 Q5: (15 points)

By Fourier - Poisson formula

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{4kt}} f(y)\, dy$$

$$= \int_0^\infty \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{4kt}} e^{-y}\,dy$$

$$= \int_0^\infty \frac{1}{\sqrt{4\pi kt}} e^{\frac{-x^2-y^2+2xy-4kty}{4kt}}\,dy$$

Notice $-x^2-y^2+2xy-4kty$

$$= -x^2-y^2+2xy-4kty-4k^2t^2+4ktx+4k^2t^2-4ktx$$

$$= -(y+2kt-x)^2+4k^2t^2-4ktx$$

So $u(x,t) = \int_0^\infty \frac{1}{\sqrt{4\pi kt}} e^{\frac{-(y+2kt-x)^2+4k^2t^2-4ktx}{4kt}}\,dy$

$$= \frac{e^{\frac{4k^2t^2-4ktx}{4kt}}}{\sqrt{4\pi kt}} \int_0^\infty e^{-\frac{(y+2kt-x)^2}{4kt}}\,dy$$

use change of variables $s = \frac{y+2kt-x}{\sqrt{4kt}}$

$$= \frac{e^{kt-x}}{\sqrt{4\pi kt}} \int_{\frac{2kt-x}{\sqrt{4kt}}}^\infty \sqrt{4kt}\, e^{-s^2}\,ds$$

$$= \frac{e^{kt-x}}{\sqrt{\pi}} \int_{\frac{2kt-x}{\sqrt{4kt}}}^\infty e^{-s^2}\,ds$$

As $t \to \infty$ we have $\frac{2kt-x}{\sqrt{4kt}} \to \infty$ and

$$\int_{\frac{2kt-x}{\sqrt{4kt}}}^\infty e^{-s^2} \to 0 .$$

on the other hand, we have $e^{kt-x} \to \infty$ as $t \to \infty$.

so we will use the L'Hopital's rule to calculate limit.

$$\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty}\left[\frac{e^{kt-x}}{\sqrt{\pi}}\int_{\frac{2kt-x}{\sqrt{4kt}}}^{\infty} e^{-s^2}\,ds\right]$$

$$= \lim_{t\to\infty}\frac{\int_{\frac{2kt-x}{\sqrt{4kt}}}^{\infty} e^{-s^2}\,ds}{\sqrt{\pi}\cdot e^{-kt+x}}$$

L'Hopital $$= \lim_{t\to\infty}\frac{e^{-\frac{(2kt-x)^2}{4kt}}\cdot\frac{\partial}{\partial t}\left(\frac{2kt-x}{\sqrt{4kt}}\right)}{\sqrt{\pi}\cdot(-k)e^{-kt+x}}$$

$$= \lim_{t\to\infty}\frac{e^{-\frac{4k^2t^2+x^2-4ktx}{4kt}}\cdot\frac{\partial}{\partial t}\left(\sqrt{k}\sqrt{t}-\frac{x}{\sqrt{4k}}\cdot\frac{1}{\sqrt{t}}\right)}{-k\sqrt{\pi}\,e^{-kt+x}}$$

$$= \lim_{t\to\infty}\frac{e^{-kt-\frac{x^2}{4kt}+x}\cdot\left(\frac{\sqrt{k}}{2\sqrt{t}}+\frac{x}{4\sqrt{k}\cdot t\sqrt{t}}\right)}{-k\sqrt{\pi}\,e^{-kt+x}}$$

$$= \lim_{t\to\infty}\frac{e^{-\frac{x^2}{4kt}}\left(\frac{\sqrt{k}}{2\sqrt{t}}+\frac{x}{4\sqrt{k}\cdot t\sqrt{t}}\right)}{-k\sqrt{\pi}}$$

Since $e^{-\frac{x^2}{4kt}} < e^0 < 1$

and $\lim_{t\to\infty}\frac{\sqrt{k}}{2\sqrt{t}} = 0$, $\lim_{t\to\infty}\frac{x}{4\sqrt{k}\cdot t\sqrt{t}} = 0$

we have $\lim_{t\to\infty} u(x,t) = 0$