# Selected solutions to CW 10

1. By Fourier-Poisson formula.

$$u(x,t) = \int_{-\infty}^{\infty} K(x-y,t)\, f(y)\, dy \quad \text{with } f \equiv 1$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\kappa t}} e^{-\frac{|x-y|^2}{4\kappa t}} \cdot 1\, dy$$

change of variables $\tilde{y} = x - y$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\kappa t}} e^{-\frac{\tilde{y}^2}{4\kappa t}}\, d\tilde{y}$$

$$= 1$$

the temperature for all later time is 1, the same as initial temperature.

2. By F-P formula.

$$u(x,t) = \int_{-\infty}^{\infty} K(x-y,t)\, f(y)\, dy$$

By the definition of $f(y)$.

$$= \int_{-L}^{L} \frac{1}{\sqrt{4\pi\kappa t}} e^{-\frac{|x-y|^2}{4\kappa t}} \cdot 1 \cdot dy$$

Doing a change of variable

$$s = \frac{x-y}{\sqrt{4\kappa t}}, \qquad -\sqrt{4\kappa t}\cdot ds = dy$$

we get

$$u(x,t) = \int_{\frac{x+L}{\sqrt{4\kappa t}}}^{\frac{x-L}{\sqrt{4\kappa t}}} \frac{1}{\sqrt{4\pi\kappa t}} e^{-s^2} \left(-\sqrt{4\kappa t}\right) ds$$

$$= \int_{\frac{x-L}{\sqrt{4kt}}}^{\frac{x+L}{\sqrt{4kt}}} \frac{1}{\sqrt{\pi}} e^{-s^2} ds$$

so
$$\lim_{t\to\infty} u(x,t) = \int_{\lim_{t\to\infty}\frac{x-L}{\sqrt{4kt}}}^{\lim_{t\to\infty}\frac{x+L}{\sqrt{4kt}}} \frac{1}{\sqrt{\pi}} e^{-s^2} ds$$

$$= \int_0^0 \frac{1}{\sqrt{\pi}} e^{-s^2} ds$$

$$= 0$$

4. By F-P formula,

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{4kt}} e^{3y} dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{4kt}+3y} dy$$

Notice
$$-\frac{(x-y)^2}{4kt}+3y = -\frac{x^2-2xy+y^2-12kty}{4kt}$$

$$= -\frac{(y-x-6kt)^2}{4kt}+9kt+3x$$

So
$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(y-x-6kt)^2}{4kt}+9kt+3x} dy$$

$$= e^{9kt+3x} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(y-x-6kt)^2}{4kt}} dy$$

Do change of variables

$$s = \frac{y-x-6k}{\sqrt{4kt}}, \quad \sqrt{4kt}\, ds = dy$$

We get

$$u(x,t) = e^{9kt+3x} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-s^2} \cdot \sqrt{4kt}\, ds$$

$$= e^{9kt+3x} \left[ \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-s^2} ds \right]$$

$$= e^{9kt+3x} \left[ \frac{1}{\sqrt{\pi}} \cdot \sqrt{\pi} \right]$$

$$= e^{9kt+3x} \to +\infty$$

as $t \to \infty$

Notice

$$\int_{-\infty}^{\infty} e^{-s^2} ds = \sqrt{\pi}$$