Queen Mary University of London

Main Examination period 2022 – January – Semester A

MTH5104: Convergence and Continuity

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

All work should be **handwritten** and should **include your student number**.

The exam is available for a period of **24 hours**. Upon accessing the exam, you will have **3 hours** in which to complete and submit this assessment.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: M. Jerrum, S. Majid

You may assume any standard properties of the functions sin(), cos(), exp() and ln(), including the fact that they are continuous.

Question 1 [20 marks]. In this question, $A, B \subseteq \mathbb{R}$ are non-empty sets of real numbers.

(a) Define what it means for $x \in \mathbb{R}$ to be an **upper bound** for A, and for it to be the **supremum** of A. **[4]**

(b) For each of the specifications below, either give an example of a set A that meets the specification, or state that no such set exists. No justification is required. **[8]**

(i) A countable set that is not bounded above.

(ii) A countable set that is bounded above but has no maximum.

(iii) An uncountable set that is bounded above but has no supremum.

(iv) An uncountable set that has a maximum.

(v) A finite set that has no maximum.

(c) Suppose A has no maximum. Starting with the definition of maximum of a set, prove the following statement:

$$\forall x \in A\ \exists y \in A : y > x.$$ **[4]**

(d) Suppose B has the property that $|x - y| < 1$ for all $x, y \in B$. Prove that B has a supremum. **[4]**

Solution:

(a) $x \in \mathbb{R}$ is a upper bound for A if $y \leq x$ for all $y \in A$. $x \in \mathbb{R}$ is the supremum of A if x is an upper bound for A and all other upper bounds z satisfy $z \geq x$.

(b) (i) $\mathbb{N}$.

(ii) $\{-1/n : n \in \mathbb{N}\}$.

(iii) Set does not exist.

(iv) $[0, 1]$.

(v) Set does not exist.

(c) From (a), x is the maximum of A if $x \in A$ and $\forall y \in A : y \leq x$. That A **does** have a maximum is therefore expressed by $\exists x \in A\ \forall y \in A : y \leq x$. Negating this expression yields $\forall x \in A\ \exists y \in A : y > x$.

(d) Select any $y \in B$. (We are told B is non-empty.) Any $x \in B$ satisfies $|x - y| < 1$. Thus, B is bounded above by $y + 1$. By the completeness axiom, B has a supremum.

Comments:

(a) bookwork; (b) examples of (i)–(iv) have appeared in the course, and (v) is used regularly; 1 mark each for exists or not, and 1 extra mark for the examples in (i), (ii) and (iv); (c) negating quantified formulas is a topic in the preamble to the module; (d) unseen.

Question 2 [20 marks]. In this question $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$, $(u_n)_{n=1}^{\infty}$ and $(v_n)_{n=1}^{\infty}$ are **sequences** of real numbers.

(a) Define (using a quantifier expression) what it means for $(x_n)_{n=1}^{\infty}$ to **converge to** $x \in \mathbb{R}$. **[2]**

(b) Suppose (x_n) converges to $x \in \mathbb{R}$ and that $a, b \in \mathbb{R}$ are real numbers with $a > 0$. Define the sequence $(y_n)_{n=1}^{\infty}$ by $y_n = ax_n + b$ for all $n \in \mathbb{N}$. Prove, **directly from the definition of a convergent sequence**, that (y_n) converges to $ax + b$. **[8]**

(c) Assume now that $(u_n)_{n=1}^{\infty}$ is an increasing sequence. Define the sequence $(v_n)_{n=1}^{\infty}$ by $v_n = \min\{u_n, 1\}$ for all $n \in \mathbb{N}$. Prove that (v_n) converges to a limit. You may appeal to any of the results covered in the course, provided you indicate which you are using. **[6]**

(d) Suppose (u_n) and (v_n) are as in part (c). If (u_n) converges to u and (v_n) to v, what relationship holds between u and v, and why? **[4]**

Solution:

(a) $(x_n)_{n=1}^{\infty}$ converges to x iff

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n > N\ :\ |x_n - x| < \varepsilon.$$

(b) Let $\varepsilon > 0$ be arbitrary. Since (x_n) converges to x, there exists $N \in \mathbb{N}$ such that $|x_n - x| < \varepsilon/a$ for all $n > N$. Then

$$|y_n - (ax + b)| = |(ax_n + b) - (ax + b)| = |ax_n - ax| = a|x_n - x| < \varepsilon$$

for all $n > N$. Thus, (y_n) converges to $ax + b$.

(c) The sequence (v_n) is increasing. (If $u_{n+1} > 1$ then $v_n \leq 1 = v_{n+1}$; and if $u_{n+1} \leq 1$ then $v_n \leq u_n \leq u_{n+1} = v_{n+1}$.) It is also bounded above by 1. An increasing sequence that is bounded above is necessarily convergent.

(d) From $v_n = \min\{u_n, 1\}$ we deduce that $v_n \leq u_n$ for all $n \in \mathbb{N}$. It follows from a result in the course that if (u_n) and (v_n) converge to u and v, respectively, then $v \leq u$.

Comments:
(a) bookwork; (b) a simpler version of the sum of two sequences, which is in the module; (c), (d) unseen.

Question 3 [20 marks]. For each of the **sequences** $(x_n)_{n=1}^{\infty}$ defined in parts (a)–(e), decide if the sequence converges and, if so, to what value. Justify your answers. You may appeal to any of the results covered in the course, provided you indicate which you are using.

(a) $x_n = \dfrac{n+1}{n^2}(1+\cos(n\pi/2))$. **[4]**

(b) $x_n = \dfrac{n+1}{n}(1+\cos(n\pi/2))$. **[4]**

(c) $x_n = \dfrac{1}{n} + \dfrac{n+1}{n}(1+\cos(n\pi/2))$. **[4]**

(d) $x_n = \dfrac{a\sqrt{n}+b}{c\sqrt{n}+d}$, where $a, b, c, d \in \mathbb{R}$ satisfy $c > 0$ and $d \geq 0$. **[4]**

(e) $x_n = \exp\left(\dfrac{n}{n+1}\right)$. **[4]**

Solution:

(a) Converges to 0. The sequence $(1/n)$ converges to 0 (building block in the module) and so does $(2/n)$. Since $|(n+1)/n^2| \leq |2n/n^2| = |2/n|$, the sequence $((n+1)/n^2)$ converges to 0 by Dominated Convergence. Note that $|1+\cos(n\pi/2)| \leq 2$. The product $(y_n z_n)$ of a sequence (y_n) converging to 0 and a bounded sequence (z_n) is a sequence converging to 0.

(b) Does not converge. Observe that $|x_{4k+2} - x_{4k+1}| = |0 - (4k+2)/(4k+1)| > 1$. If (x_n) converged, then the sequence of differences $\big(|x_{n+1} - x_n|\big)$ would converge to 0.

(c) Does not converge. Note that this is the sum of a convergent sequence $(1/n)$ and the non-convergent series from (b). Now use the contrapositive of the fact that the difference of two convergent sequences is convergent.

(d) Converges to a/c. Divide through by $\sqrt{n}$ to get $x_n = \frac{a+b/\sqrt{n}}{c+d/\sqrt{n}}$. The sum $(y_n + z_n)$ and quotient (y_n/z_n) of a sequence y_n converging to y and a sequence (z_n) converging to z is a sequence converging to $y+z$ and y/z, respectively. (In the latter case, (y_n/z_n) must be defined and $z \neq 0$.) We saw in the course that the sequence $(1/\sqrt{n})$ converges to 0. So the numerator converges to a and the denominator to c. So the quotient converges to a/c. (Note that the denominators are never 0.)

(e) Converges to e. The sequence $(1+1/n)$ converges to 1 (the sum of two convergent sequences); also, $\exp(\cdot)$ is a continuous function, so $\big(\exp((n+1)/n)\big)$ converges to $\exp(1) = e$.

Comments:

These precise sequences have not appeared in the course (unless by accident), but similar ones have. In each case, 2 marks for the right answer without justification. I will accept "by Theorem 2.19", etc., as acceptable justification.

Question 4 [20 marks].
Parts (a)–(c) of this question concern the **power series** $\sum_{k=1}^{\infty} x^k/k$. In these parts, you may appeal to any of the results covered in the module, provided you indicate which you are using.

(a) Prove that $\sum_{k=1}^{\infty} x^k/k$ converges absolutely when $|x| < 1$. **[4]**

(b) Prove that $\sum_{k=1}^{\infty} x^k/k$ does not converge when $|x| > 1$. **[6]**
(You may use the fact that $(1+\alpha)^k \geq \alpha k$ for all $k \in \mathbb{N}$ and $\alpha > 0$.)

(c) State whether $\sum_{k=1}^{\infty} x^k/k$ converges when $x = 1$ and $x = -1$ (two cases). No justification is required. **[4]**

(d) For each of the power series (i)–(iii) below, state whether it converges only at $x = 0$, converges for all $x \in \mathbb{R}$, or converges for some non-zero x but not all (in which case, state the radius of convergence). No justification is required.

$$\text{(i)} \sum_{k=1}^{\infty} 2^k k!\, x^k \qquad \text{(ii)} \sum_{k=1}^{\infty} 2^k x^k \qquad \text{(iii)} \sum_{k=1}^{\infty} \frac{2^k}{k!} x^k.$$ **[6]**

Solution:

(a) We have $0 \leq |x^k/k| \leq |x|^k$ for all $k \in \mathbb{N}$. The geometric series $\sum_{k=1}^{\infty} c^k$ converges when $|c| < 1$ (result from the module). Thus $\sum_{k=1}^{\infty} |x^k/k|$ converges, by the comparison test, and $\sum_{k=1}^{\infty} x^k/k$ converges absolutely.

(b) Let $\alpha = |x| - 1 > 0$. We have

$$|x^k/k| = |x|^k/k = (1+\alpha)^k/k \geq \alpha k/k = \alpha.$$

Hence (x^k/k) does not converge to zero. Hence the series $\sum_{k=1}^{\infty} x^k/k$ cannot converge.

(c) Does not converge in the case $x = 1$ (harmonic series) and does converge when $x = -1$ (alternating harmonic series).

(d) (i) converges only when $x = 0$, (ii) has radius of convergence $\frac{1}{2}$, and (iii) converges for all $x \in \mathbb{R}$.

Comments:
(a), (b) unseen; (c) standard examples from the lectures; (d) barely disguised (by the inclusion of 2^k) examples from the lectures.

Question 5 [20 marks].

(a) Define (using a quantifier expression) what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be **continuous** at a point $a \in \mathbb{R}$. **[3]**

(b) Prove **directly from the definition** that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 - x + 2$ is continuous at each point $a \in \mathbb{R}$.

[Given ε, you may like to try δ of the form $\delta = \min\{c\varepsilon, 1\}$ for a suitable constant $c > 0$.] **[8]**

(c) How do you know that the function $g(x) = \exp(x^2 - x + 2)$ is continuous at all points in $\mathbb{R}$? **[3]**

(d) Using the Intermediate Value Theorem (IVT), show that the equation $x^2 - x + 2 = \exp(x)$ has a solution for x in the interval $[0, 1]$. Show explicitly that the conditions of the IVT are satisfied. **[6]**

Solution:

(a) f is continuous at a if

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall h \in \mathbb{R}, |h| < \delta \;:\; |f(a+h) - f(a)| < \varepsilon.$$

(b) We have

$$\begin{aligned} |f(a+h) - f(a)| &= |((a+h)^2 - (a+h) + 2) - (a^2 - a + 2)| \\ &= |2ah + h^2 - h| \leq (|2a| + 1)|h| + |h|^2. \end{aligned}$$

Given $\varepsilon > 0$, set $\delta = \min\{c\varepsilon, 1\}$ (following the hint). Then for all h with $|h| < \delta$ we have

$$|f(a+h) - f(a)| < (2|a| + 1)\delta + \delta^2 \leq (2|a| + 2)\delta.$$

using $\delta \leq 1$. Now set $c = 1/(2|a| + 2)$ and note that $(2|a| + 2)\delta \leq \varepsilon$.

(c) The function g is the composition $g = \exp \circ f$ of two continuous functions: we are told the exp is continuous, and f is continuous by part (b).

(d) Consider the function $h(x) = f(x) - \exp x$. The function h is the difference of two continuous functions and hence continuous. Let $a = 0$ and $b = 1$. Then $h(a) = 1 \geq 0 \geq 1 - e = h(b)$. By the IVT, h has a root in $[0, 1]$. At this root, $x^2 - x + 2 = \exp(x)$.

Comments:

(a) Bookwork; (b) $f(x) = x^2$ seen in lectures (and a more general quadratic is among the exercises); (c) is an easy application of a result from the module; (d) is similar to examples in the module.

End of Paper.