Queen Mary University of London

Main Examination period 2024 – May/June – Semester B

MTH5104: Convergence and Continuity (Practice Exam)

Examiners: N. Nabijou

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You will have a period of **3 hours** to complete the exam and submit your solutions.

You should attempt ALL questions. Marks available are shown next to the questions.

The exam is closed-book, and **no outside notes are allowed**.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Exam papers must not be removed from the examination room.

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Important: All your answers must be justified. Unless the question explicitly indicates otherwise, you may use any result from the lectures, provided you state the result clearly.

Question 1 [25 marks].

(a) Given a subset $A \subseteq \mathbb{R}$ and a real number $a \in \mathbb{R}$, define what it means for a to be the supremum of A. [**5**]

(b) Consider the following set of real numbers:

$$A = \left\{ \frac{3n^2 + n + 3}{n^3 + n} : n \in \mathbb{N} \right\} \subseteq \mathbb{R}.$$

(i) Prove that A is bounded. [**4**]

(ii) Prove that $\inf(A) = 0$. Can you replace inf by min? Justify your answer. [**4**]

(iii) Find $\sup(A)$ and $\max(A)$, justifying your answer in each case. [**3**]

(b) Let $B \subseteq \mathbb{R}$ be nonempty and bounded.

(i) Prove that $\inf(B) \leq \sup(B)$. [**5**]

(ii) Does there exist a B such that $\inf(B) = \sup(B)$? [**4**]

Solutions to Question 1.

(a) We say that a is the supremum of A if and only if the following two conditions hold:

- a is an upper bound for A: $\forall x \in A : x \leq a$.
- a is the least upper bound for A: $\forall z < a, \exists x \in A : z < x$. **[5]**

(b) Consider the set of real numbers

$$A = \left\{ \frac{3n^2 + n + 3}{n^3 + n} : n \in \mathbb{N} \right\} \subseteq \mathbb{R}.$$

(i) We must show that A is bounded. We begin by simplifying the expression

$$\frac{3n^2 + n + 3}{n^3 + n} = \frac{3n^2 + 3}{n^3 + n} + \frac{n}{n^3 + n} = \frac{3(n^2 + 1)}{n(n^2 + 1)} + \frac{1}{n^2 + 1} = \frac{3}{n} + \frac{1}{n^2 + 1}.$$

Both of the sequences

$$\frac{3}{n} \quad \text{and} \quad \frac{1}{n^2 + 1}$$

are decreasing. Hence, their sum is also decreasing. It follows that A is bounded above by the $n = 1$ term, which is $7/2$. On the other hand all the terms are strictly positive, so A is bounded below by 0. We conclude that A is bounded. **[4]**

(ii) We have shown in the previous part that 0 is a lower bound for A. To show that it is the greatest lower bound, we need to show that for every $\epsilon > 0$ there exists an $n \in \mathbb{N}$ such that

$$\frac{3n^2 + n + 3}{n^3 + n} < \epsilon.$$

For $n \in \mathbb{N}$ we have

$$\frac{3n^2 + n + 3}{n^3 + n} \leq \frac{3n^2 + n + 3}{n^3} = \frac{3}{n} + \frac{1}{n^2} + \frac{3}{n^3} \leq \frac{3}{n} + \frac{1}{n} + \frac{3}{n} = \frac{7}{n}.$$

So given $\epsilon > 0$ we choose $n \in \mathbb{N}$ such that $n > 7/\epsilon$ (this exists by the Archimedean principle). Then $7/n < \epsilon$ as required.

We cannot replace inf by min. A result from lectures states that $\min(A)$ exists if and only if $\inf(A)$ exists and belongs to A. In this case $\inf(A) = 0$ exists but does not belong to A, since every element of A is strictly positive. **[4]**

(iii) We have already shown that the sequence is decreasing. Hence, $\sup(A) = \max(A)$ is equal to the $n = 1$ term, which is $7/2$. **[3]**

(c) (i) Since B is nonempty there exists an element $b \in B$. Then by definition we have

$$\inf(B) \leq b \leq \sup(B)$$

from which we conclude $\inf(B) \leq \sup(B)$. **[5]**

(ii) Equality can hold if B is a singleton: e.g. $B = \{1\}$ has $\inf(B) = \sup(B) = 1$. **[4]**

Question 2 [25 marks].

(a) Define what it means for a sequence (x_n) to converge to a value $x \in \mathbb{R}$. [**5**]

(b) Let (x_n) and (y_n) be sequences, and suppose that $x_n \to \infty$ and $y_n \to \infty$. Prove that $x_n + y_n \to \infty$. [**6**]

(c) Give an example of a sequence that contains both a bounded subsequence and an unbounded subsequence. [**5**]

(d) For each of the following sequences, decide whether or not it converges, and justify your answer.
(You may use any result from the lectures, but you must state the result clearly.)

(i) $x_n = \dfrac{1}{n^{2023}}$. [**3**]

(ii) $x_n = \dfrac{1}{n + n^2}$. [**3**]

(iii) $x_n = n(2 + \sin(n\sqrt{2023}))$. [**3**]

Solutions to Question 2.

(a) We say that (x_n) converges to x if and only if:

$$\forall \epsilon > 0 \; \exists N \in \mathbb{N} \; \forall n > N : |x_n - x| < \epsilon. \qquad [\mathbf{5}]$$

(b) Fix $R > 0$. We must show that there exists an $N \in \mathbb{N}$ such that for $n > N$ we have $x_n + y_n > R$. Since $x_n \to \infty$ there exists $N_1 \in \mathbb{N}$ such that for $n > N_1$ we have

$$x_n > R/2.$$

Similarly since $y_n \to \infty$ there exists $N_2 \in \mathbb{N}$ such that for $n > N_2$ we have

$$y_n > R/2.$$

Set $N = \max\{N_1, N_2\}$. Then for $n > N$ we have $n > N_1$ and $n > N_2$ and so

$$x_n + y_n > R/2 + R/2 = R. \qquad [\mathbf{6}]$$

(c) Define

$$x_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd.} \end{cases}$$

Then the subsequence (x_{2k}) is bounded (since $x_{2k} = 0$ for all $k \in \mathbb{N}$), but the subsequence (x_{2k-1}) is unbounded (since $x_{2k-1} = 2k - 1$ for all $k \in \mathbb{N}$). [**5**]

(d) (i) Consider the sequence

$$x_n = \frac{1}{n^{2023}}.$$

We claim that $x_n \to 0$. Given $\epsilon > 0$ take $N \in \mathbb{N}$ such that $N > 1/\epsilon$ (this exists by the Archimedean property). Then for $n > N$ we have

$$\left| \frac{1}{n^{2023}} \right| = \frac{1}{n^{2023}} \leq \frac{1}{n} < \frac{1}{N} < \epsilon$$

as required. [**3**]

(ii) Consider the sequence

$$x_n = \frac{1}{n + n^2}.$$

We have

$$|x_n| = \frac{1}{n + n^2} \leq \frac{1}{n}$$

and we have seen in lectures that $\frac{1}{n} \to 0$. A result from lectures says that if $(x_n), (y_n)$ are sequences with $y_n \to 0$ and $|x_n| \leq |y_n|$ for all $n \in \mathbb{N}$, then $x_n \to 0$. Applying this in our case with $y_n = \frac{1}{n}$, we conclude that $x_n \to 0$. [**3**]

(iii) Consider the sequence

$$x_n = n(2 + \sin(n\sqrt{2023})).$$

We claim that $x_n \to \infty$, which in particular implies that (x_n) does not converge. Since $\sin(n\sqrt{2023}) \in [-1, 1]$ we have $2 + \sin(n\sqrt{2023}) \in [1, 3]$ and so

$$x_n = n(2 + \sin(n\sqrt{2023})) \geq n.$$

Fix $R > 0$. Take $N \in \mathbb{N}$ with $N > R$. Then for $n > N$ we have

$$x_n \geq n > N > R$$

as required. **[3]**

Question 3 [25 marks].

(a) Define what it means for a series to be conditionally convergent. [**6**]

(b) For each of the following series, decide whether or not it converges. You do not need to calculate its value.

(i) $\displaystyle\sum_{k=1}^{\infty} \frac{2^k + 3^k}{2^k + 7^k}$. [**6**]

(ii) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2 + \sqrt{k} + 1}$. [**6**]

(c) Suppose we are given two conditionally convergent series $\sum_{k=1}^{\infty} x_k$ and $\sum_{k=1}^{\infty} y_k$. Does it follow that the series

$$\sum_{k=1}^{\infty} (x_k + y_k)$$

is conditionally convergent? Prove or give a counterexample. [**7**]

Solutions to Question 3.

(a) A series $\Sigma_{k=1}^{\infty} x_k$ is conditionally convergent if it is convergent but not absolutely convergent. This means that $\Sigma_{k=1}^{\infty} x_k$ exists but $\Sigma_{k=1}^{\infty} |x_k|$ does not exist. **[6]**

(b) (i) We use the ratio test. Let $x_k = (2^k + 3^k)/(2^k + 7^k)$. Then we calculate:

$$\begin{aligned} \frac{x_{k+1}}{x_k} &= \frac{2^{k+1} + 3^{k+1}}{2^{k+1} + 7^{k+1}} \cdot \frac{2^k + 7^k}{2^k + 3^k} \\ &= \frac{2^{k+1} + 3^{k+1}}{2^k + 3^k} \cdot \frac{2^k + 7^k}{2^{k+1} + 7^{k+1}} \\ &= \frac{2(\frac{2}{3})^k + 3}{(\frac{2}{3})^k + 1} \cdot \frac{(\frac{2}{7})^k + 1}{2(\frac{2}{7})^k + 7} \\ &\to \frac{3}{1} \cdot \frac{1}{7} \\ &= \frac{3}{7} \\ &< 1. \end{aligned}$$

The limit is calculated using the following facts from lectures: $r \to 0$ if $|r| < 1$, and limits are compatible with addition, multiplication, and division. We conclude by the ratio test that the series converges. **[6]**

(ii) For $k \in \mathbb{N}$ we have

$$\frac{1}{k^2 + \sqrt{k} + 1} \leq \frac{1}{k^2}.$$

We have seen in lectures that the series $\Sigma_{k=1}^{\infty} \frac{1}{k^2}$ converges. The comparison test states that if (x_k) and (y_k) are sequences with $0 \leq y_k \leq x_k$, and if $\Sigma_{k=1}^{\infty} x_k$ exists, then $\Sigma_{k=1}^{\infty} y_k$ exists. Applying this to the above comparison, we conclude that the given series converges. **[6]**

(c) It does not follow. Define:

$$x_k = \frac{(-1)^k}{k}, \qquad y_k = -x_k = \frac{(-1)^{k+1}}{k}.$$

We have seen in lectures that $\Sigma_{k=1}^{\infty} x_k$ is conditionally convergent. Moreover we have seen in lectures that if we multiply a convergent series by a fixed $c \in \mathbb{R}$, the result is still convergent. Taking $c = -1$ and observing that $y_k = cx_k$, we conclude that $\Sigma_{k=1}^{\infty} y_k$ is convergent. However, it is not absolutely convergent, since $|y_k| = |x_k| = 1/k$. Thus, we have two conditionally convergent series. The sum is

$$x_k + y_k = 0$$

and so, trivially, the series $\Sigma_{k=1}^{\infty}(x_k + y_k)$ is absolutely convergent. In particular, it is not conditionally convergent. **[7]**

Question 4 [25 marks].

(a) Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ given by:

$$f(x) = 3x + 4.$$

Prove, **directly from the definition**, that $f(x)$ is continuous at all points $a \in \mathbb{R}$. [**6**]

(b) Consider the function $g\colon \mathbb{R} \to \mathbb{R}$ given by:

$$g(x) = \begin{cases} \sqrt{x} & \text{if } x > 0 \\ -1 & \text{if } x \leq 0 \end{cases}$$

Find a point $a \in \mathbb{R}$ such that $g(x)$ is not continuous at a. Justify your answer. (You may use any result from the lectures, but you must state the result clearly.) [**6**]

(c) Consider now two arbitrary functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. Suppose that $f(x)$ and $g(x)$ are both continuous at the point $a \in \mathbb{R}$. Prove **directly from the definition** that the function

$$h(x) = f(x) + g(x)$$

is also continuous at a. [**7**]

(d) Prove that there exists an $x \in \mathbb{R}$ such that $x + \cos(x) = 1$. [**6**]

Solutions to Question 4.

(a) Given $\epsilon > 0$ take $\delta = \epsilon/3$. Then for $x \in \mathbb{R}$ with $|x - a| < \delta = \epsilon/3$, we have

$$|f(x) - f(a)| = |(3x + 4) - (3a + 4)| = |3x - 3a| = 3|x - a| < 3\delta = \epsilon$$

as required. **[6]**

(b) Take $a = 0$ and suppose for a contradiction that $g(x)$ is continuous at a. Consider the sequence $x_n = 1/n$ for $n \in \mathbb{N}$. Then $x_n \to 0$. On the other hand $x_n > 0$ and so $g(x_n) = 1/\sqrt{n}$ for all $n \in \mathbb{N}$. It follows that $g(x_n) \to 0$. But on the other hand $g(0) = -1$. This contradicts a result from lectures, which states that if $g(x)$ is continuous at a and (x_n) is a sequence with $x_n \to a$, then we must have $g(x_n) \to g(a)$. **[6]**

(c) Given $\epsilon > 0$ we set $\tilde{\epsilon} = \epsilon/2$. There exists $\delta_1 > 0$ such that if $|x - a| < \delta_1$ then

$$|f(x) - f(a)| < \tilde{\epsilon}$$

and similarly there exists $\delta_2 > 0$ such that if $|x - a| < \delta_2$ then

$$|g(x) - g(a)| < \tilde{\epsilon}.$$

Take $\delta = \min\{\delta_1, \delta_2\}$. Then for $|x - a| < \delta$ we have

$$|h(x)-h(a)| = |(f(x)-f(a))+(g(x)-g(a))| \leq |f(x)-f(a)|+|g(x)-g(a)| < \tilde{\epsilon}+\tilde{\epsilon} = \epsilon$$

where the first inequality follows from the triangle inequality. **[7]**

(d) We wish to apply the Intermediate Value Theorem to the function

$$g(x) = x + \cos(x) - 1.$$

We first justify that this function is continuous. We have seen in lectures that constant functions are continuous, as are the functions x and $\cos(x)$. Moreover, continuous functions are closed under addition and multiplication. We conclude that $g(x)$ is continuous.

We wish to find $a, b \in \mathbb{R}$ such that $g(a) \leq 0$ and $g(b) \geq 0$. Then, applying the Intermediate Value Theorem will produce a point c between a and b such that $g(c) = 0$. This will be the solution to our equation.

Taking $a = -\pi$ and $b = \pi$, we compute:

$$g(-\pi) = -\pi - 2 < 0, \qquad g(\pi) = \pi - 2 > 0.$$

Applying the Intermediate Value Theorem, we obtain $c \in [-\pi, \pi]$ with $g(c) = 0$, as required. **[6]**

End of Paper.