Non-escaping set $\Lambda(f_\mu)$ is given by

$$\Lambda(f_\mu) = \{x \in [0,1] : f_\mu^n(x) \in [0,1] \ \forall n \geq 0\}$$
$$= \bigcap_{n=0}^{\infty} \Lambda_n$$

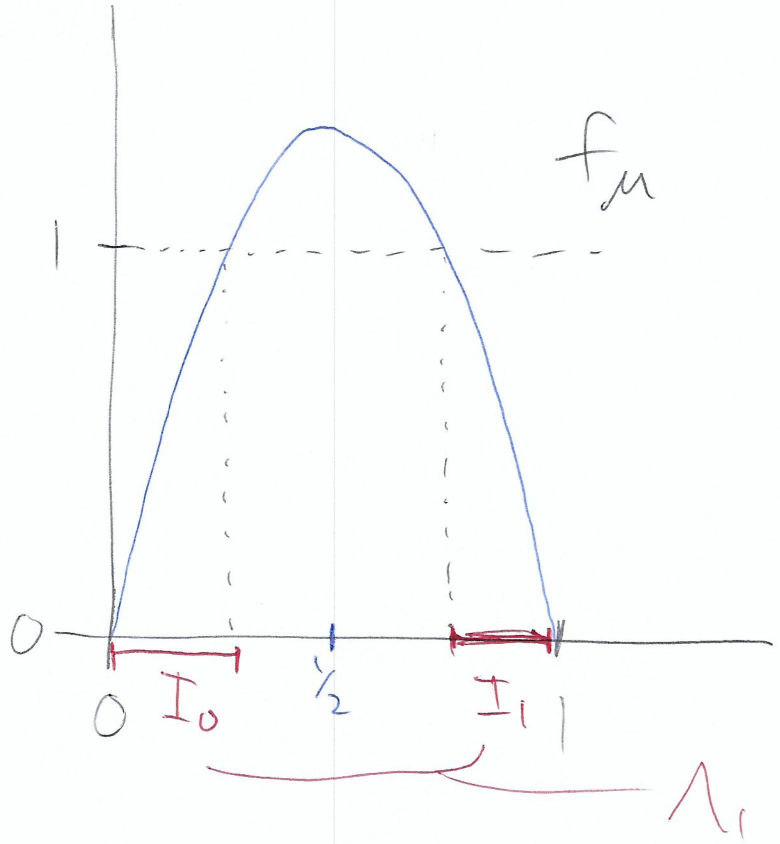
where $\Lambda_n := \{x \in [0,1] : f_\mu^n(x) \in [0,1]\}$

$= \{x \in [0,1] : f_\mu^n(x)$ is defined and belongs to $[0,1]\}$

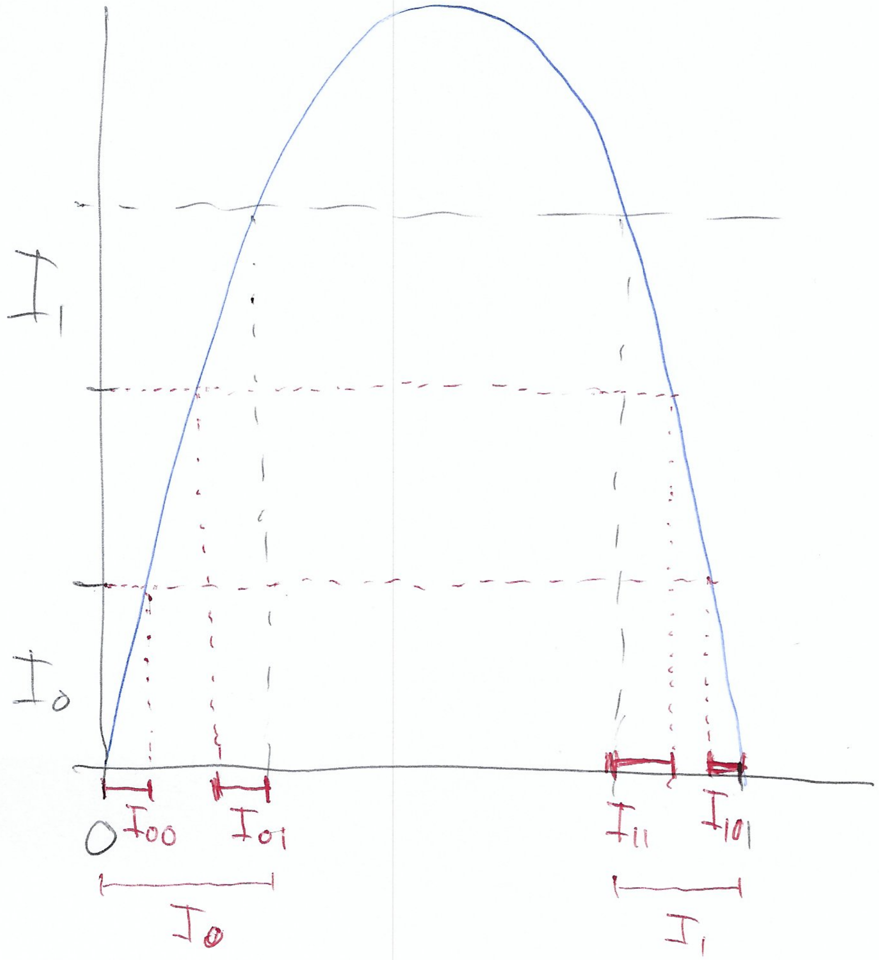
$= \{x \in [0,1] : f_\mu^i(x) \in [0,1]$ for all $0 \leq i \leq n\}$

Note that $\Lambda_0 = [0,1]$

Note that $\Lambda_1 = I_0 \cup I_1$ is the union of two disjoint closed intervals (I_0 and I_1, where $I_0 \subset [0, \tfrac{1}{2})$, $I_1 \subset [\tfrac{1}{2}, 1]$)



Note that $\Lambda_2 = I_{00} \cup I_{01} \cup I_{11} \cup I_{10}$ is the union of four disjoint closed intervals



In general, the set Λ_n can be written as a disjoint union of 2^n closed intervals

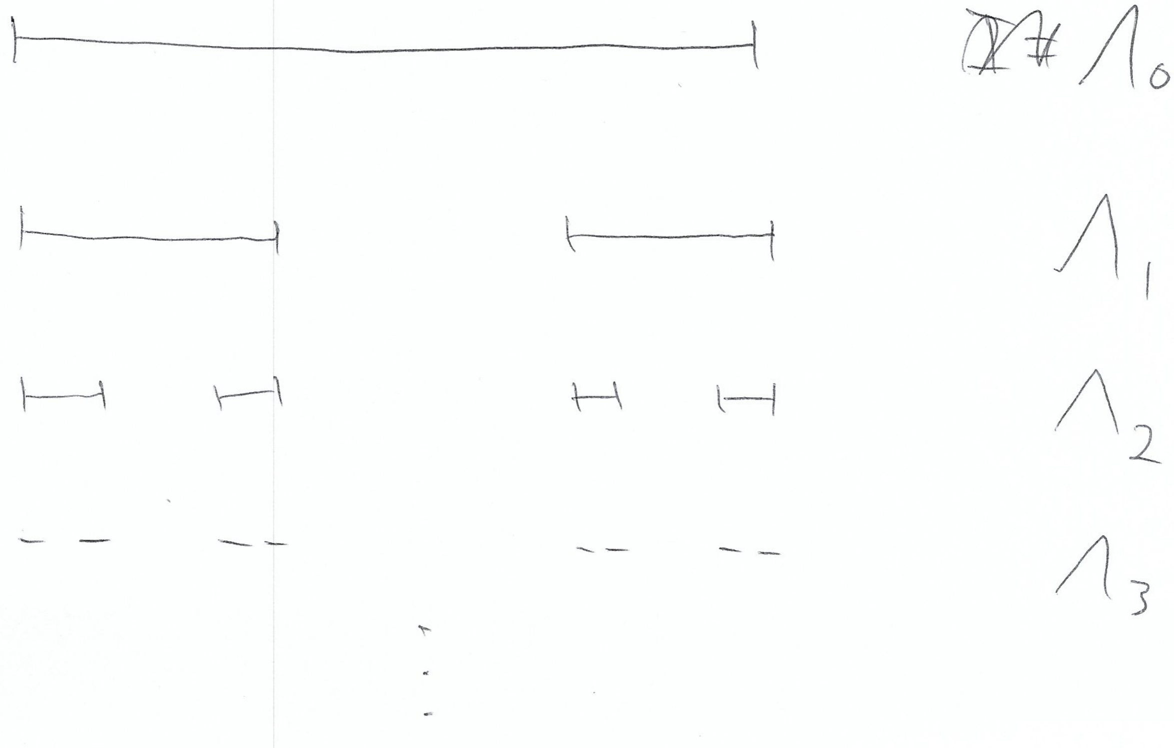
$$I_{b_1 b_2 \ldots b_n} = \{ x \in [0,1] : f_\mu^{i-1}(x) \in I_{b_i} \text{ for } 1 \leq i \leq n \}$$

where $b_i \in \{0,1\}$.

The non-escaping set $\Lambda(f_\mu)$ can be written as

$$\Lambda(f_\mu) = \bigcap_{n=0}^{\infty} \Lambda_n ,$$

and this ~~a~~ is a Cantor set



The points in $\Lambda^{=\Lambda(f_\mu)}$ correspond naturally to the set S of sequences whose entries are 0 or 1

If $b = b_1 b_2 b_3 b_4 \ldots$ is such a sequence then there is a (unique) point $x_0 \in \Lambda(f_\mu)$ such that

$$f_\mu^{i-1}(x_0) = x_{i-1} \in I_{b_i} \quad \forall i \in \mathbb{N}.$$

In fact:

<u>Prop</u>: For $\mu > 4$, the non-escaping set $\Lambda(f_\mu)$ is a Cantor set, and the dynamical system $f_\mu : \Lambda(f_\mu) \to \Lambda(f_\mu)$ is (topologically) conjugate to the shift map $\sigma : S \to S$.

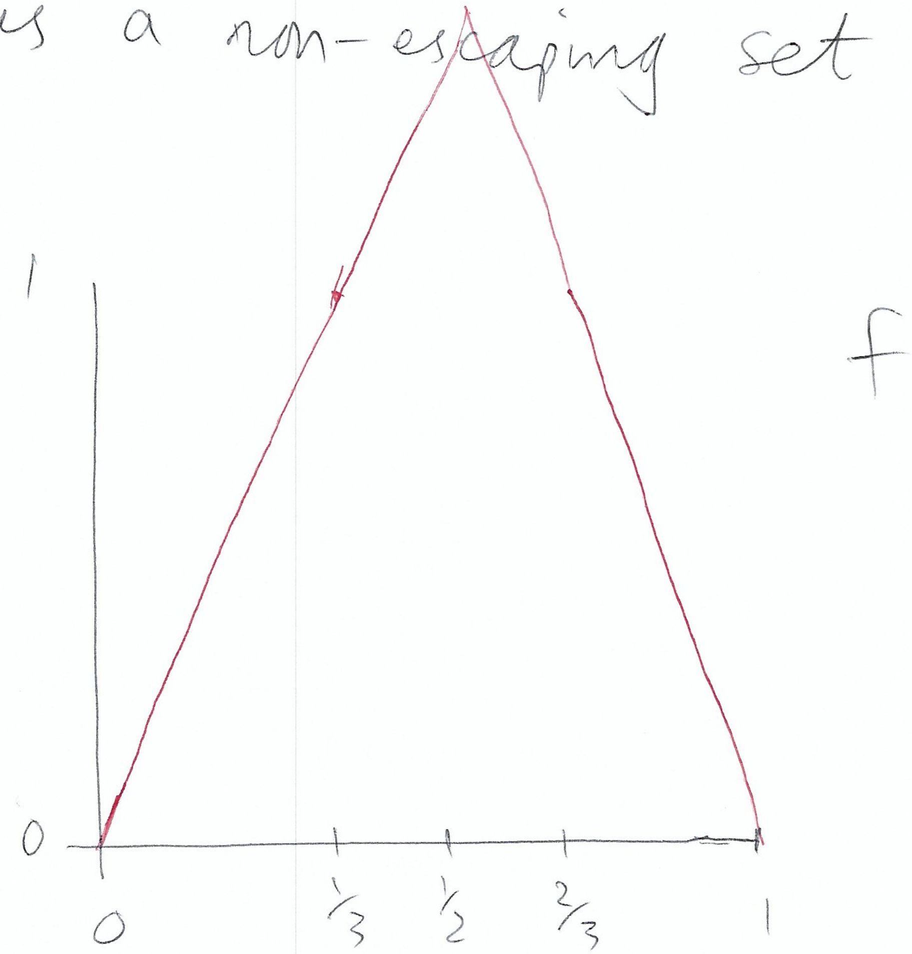
The diagram

$$\begin{tikzcd} S \arrow[r, "\sigma"] \arrow[d, "h"'] & S \arrow[d, "h"] \\ \Lambda(f_\mu) \arrow[r, "f_\mu"] & \Lambda(f_\mu) \end{tikzcd}$$

commutes (i.e. $h \circ \sigma = f_\mu \circ h$), where $h(b_1 b_2 b_3 \ldots)$ is defined to be the point $x_0 \in \Lambda(f_\mu)$ such that $x_{i-1} \in I_{b_i}$ for all $i \in \mathbb{N}$

<u>Corollary</u> For $\mu > 4$, the logistic map $f_\mu : \Lambda(f_\mu) \to \Lambda(f_\mu)$ is chaotic (in the sense of Devaney).

Example: Viewing the Middle-$\frac{1}{3}$ Cantor set as a non-escaping set



Define $f : [0,1] \longrightarrow \mathbb{R}$

$$f(x) = \begin{cases} 3x & \text{if } x \in [0, \frac{1}{2}) \\ 3 - 3x & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

Then the non-escaping set

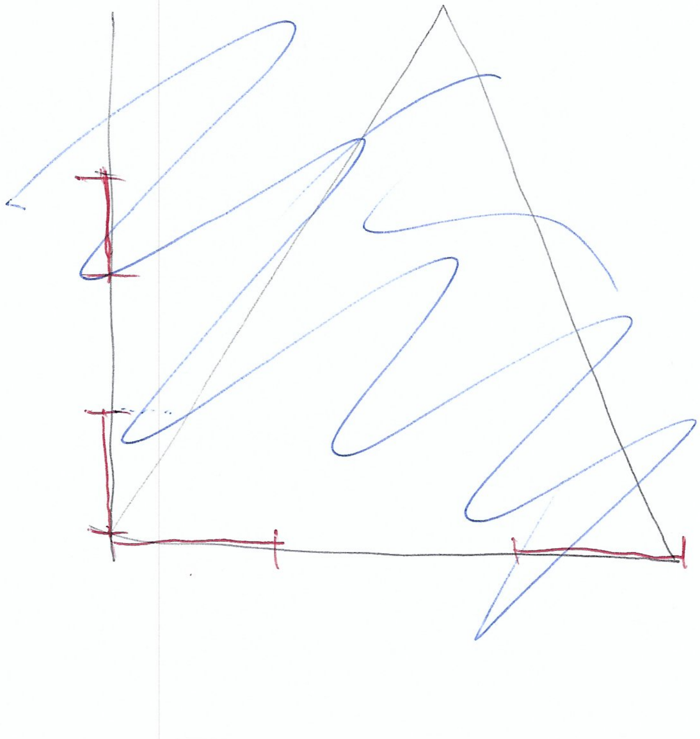
$$\Lambda(f) = \{ x \in [0,1] : f^n(x) \in [0,1] \ \forall n \geq 0 \}$$

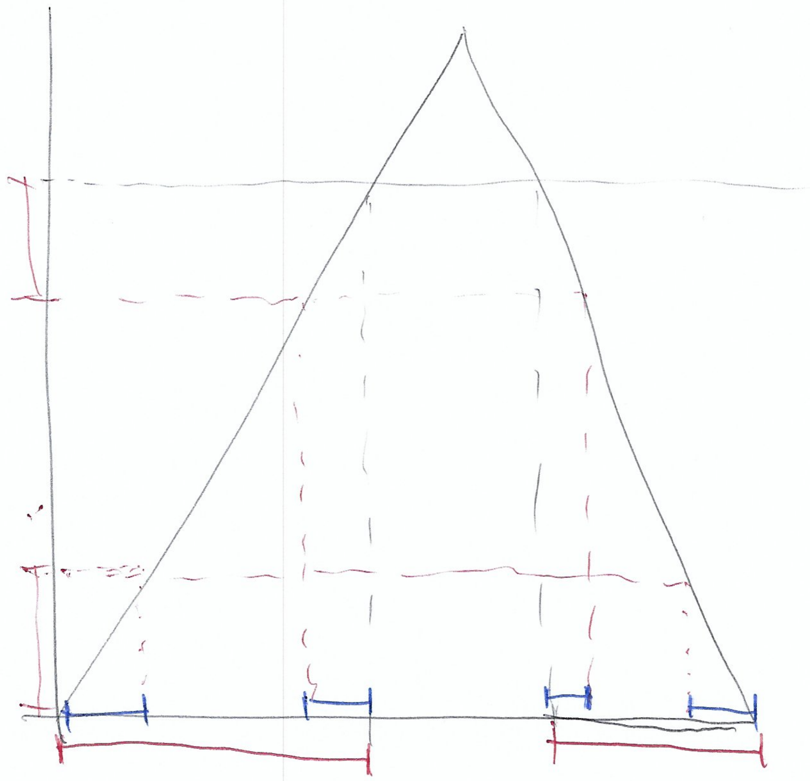
is <u>precisely</u> the middle-third Cantor set.

To see this, note that

$$\Lambda_1 = \{x \in [0,1] : f(x) \in [0,1]\}$$
$$= f^{-1}([0,1])$$
$$= [0, 1/3] \cup [2/3, 1]$$

$$\Lambda_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$$
$$= f^{-1}([0, 1/3] \cup [2/3, 1])$$





A related viewpoint

Alternatively, we could introduce maps $\varphi_1 : [0,1] \to [0,1]$ and $\varphi_2 : [0,1] \to [0,1]$ defined by $\varphi_1(x) = x/3$

and $\varphi_2(x) = \frac{x+2}{3} = x/3 + 2/3$

Note that $\varphi_1([0,1]) = [0, 1/3]$

$\varphi_2([0,1]) = [2/3, 1]$

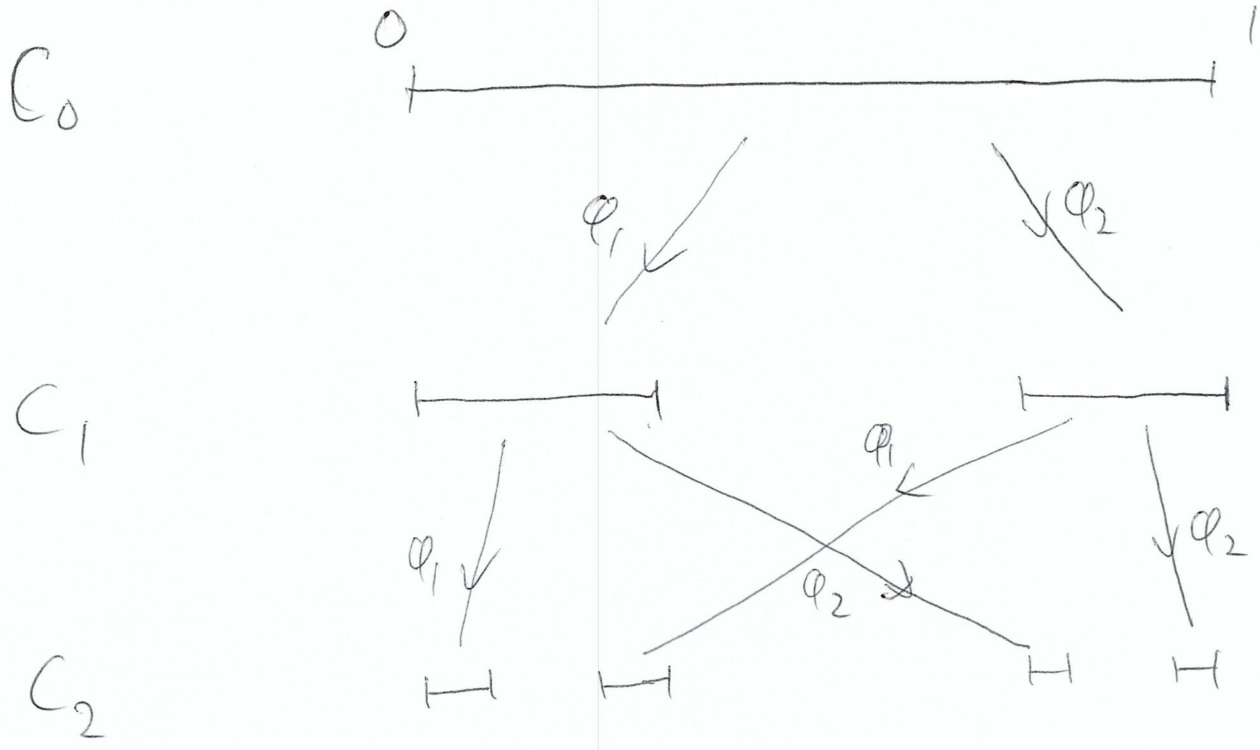
Now define a mapping Φ to act on subsets of $[0,1]$ by the formula

$$\Phi(A) = \varphi_1(A) \cup \varphi_2(A)$$

(where $\varphi_i(A) = \{\varphi_i(a) : a \in A\}$ for $i=1,2$)

Then $$\Phi([0,1]) = \varphi_1([0,1]) \cup \varphi_2([0,1])$$
$$= [0, 1/3] \cup [2/3, 1]$$

Also $$\Phi^2([0,1]) = \Phi(\Phi([0,1]))$$
$$= \Phi([0, 1/3] \cup [2/3, 1])$$
$$= \varphi_1([0,1/3] \cup [2/3,1]) \cup \varphi_2([0,1/3] \cup [2/3,1])$$
$$= [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$$



In fact $\Phi^n([0,1])$ is the set C_n in the definition of the middle-third Cantor set C.

$$\text{So } C = \bigcap_{n=0}^{\infty} C_n = \bigcap_{n=0}^{\infty} \Phi^n([0,1])$$

<u>Remark</u> We can think of C as a fixed point of Φ, ie $\Phi(C) = C$.
There is also a sense in which $\Phi^n([0,1])$ converges to C as $n \to \infty$ (this is a more advanced viewpoint).

This viewpoint suggests the following:

Defn A finite collection of maps $\varphi_1, \dots, \varphi_k$ where each $\varphi_i : [0,1] \to [0,1]$, is called an <u>iterated function system</u>.

We define an associated map Φ, acting on subsets of $[0,1]$, by

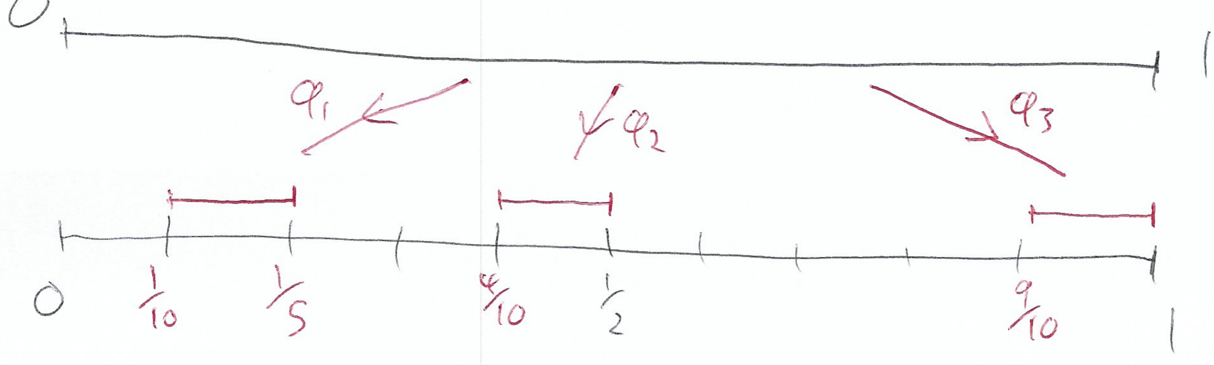
$$\Phi(A) = \bigcup_{i=1}^{k} \varphi_i(A)$$

Example Define

$$\varphi_1(x) = \frac{x+1}{10}$$

$$\varphi_2(x) = \frac{x+4}{10}$$

$$\varphi_3(x) = \frac{x+9}{10}$$



Note that $\varphi_1([0,1]) = [1/10, 2/10] =$ {numbers with decimal expansion $0.1\ldots$}

$\varphi_2([0,1]) = [4/10, 5/10] =$ {numbers with decimal expansion $0.4\ldots$}

$\varphi_3([0,1]) = [9/10, 1] =$ {numbers with decimal expansion $0.9\ldots$}

So $\Phi([0,1]) = [1/10, 2/10] \cup [4/10, 5/10] \cup [9/10, 1]$

Then $\Phi^2([0,1]) = \Phi(\Phi([0,1]))$

$= \Phi\left([1/10, 2/10] \cup [4/10, 5/10] \cup [9/10, 1]\right)$

$= \varphi_1\left([1/10, 2/10] \cup [4/10, 5/10] \cup [9/10, 1]\right)$

$\cup\ \varphi_2(\quad " \qquad " \qquad " \qquad " \quad)$

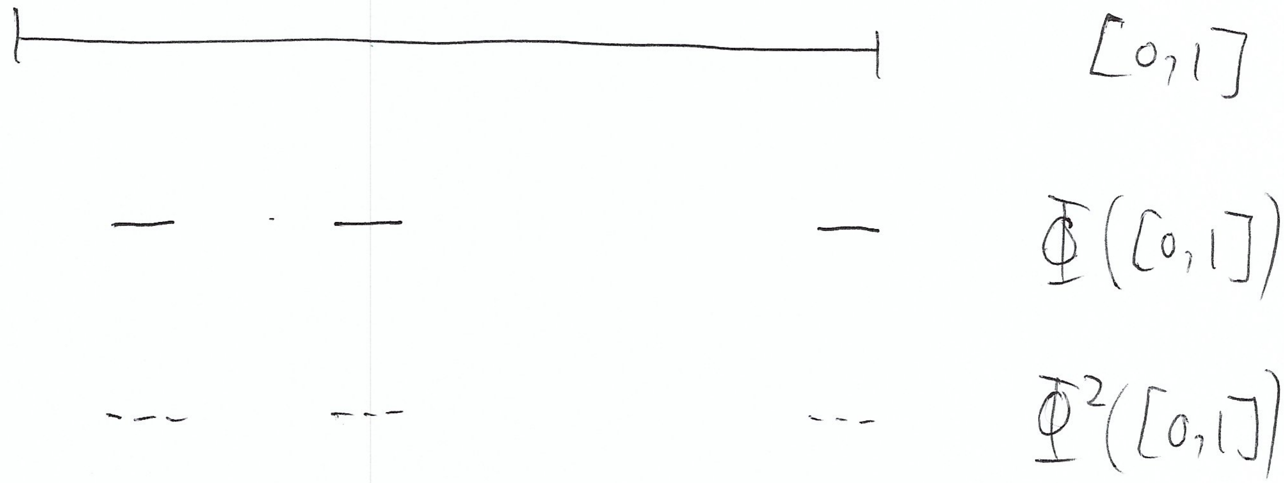
$\cup\ \varphi_3(\quad " \qquad " \qquad " \qquad " \quad)$

$= \left[\frac{11}{100}, \frac{12}{100}\right] \cup \left[\frac{14}{100}, \frac{15}{100}\right] \cup \left[\frac{19}{100}, \frac{20}{100}\right]$

$\cup \left[\frac{41}{100}, \frac{42}{100}\right] \cup \left[\frac{44}{100}, \frac{45}{100}\right] \cup \left[\frac{49}{100}, \frac{50}{100}\right]$

$\cup \left[\frac{91}{100}, \frac{92}{100}\right] \cup \left[\frac{94}{100}, \frac{95}{100}\right] \cup \left[\frac{99}{100}, \frac{100}{100}\right]$

(note that e.g. the interval $\left[\frac{14}{100}, \frac{15}{100}\right]$ is precisely the set of all numbers which admit a decimal expansion whose first two digits are 14, i.e. they have a decimal expansion $0.14\ldots\ldots$)



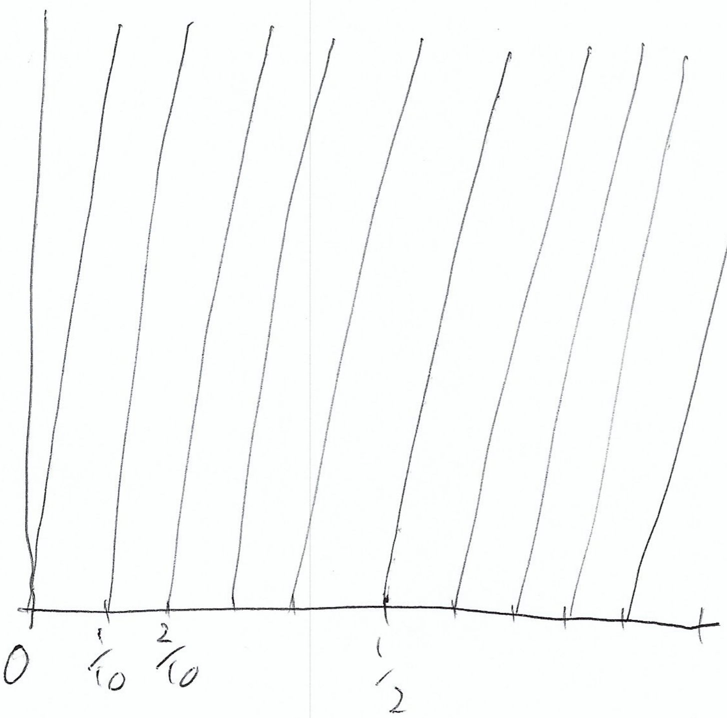
The intersection $\bigcap_{n=0}^{\infty} \Phi^n([0,1])$ is a Cantor set, let's call it K.

It is the set of all numbers in $[0,1]$ which have a decimal expansion only containing the digits 1, 4 and 9.

Also $\Phi(K) = K$.

Also, if we define $f:[0,1] \to [0,1]$ by

$$f(x) = \begin{cases} 10x \pmod 1 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$$



Then $f(K) = K$.

In view of the fact that $f(K)=K$ we might consider the restricted map $f|_K : K \to K$. This restricted map is topologically conjugate to the shift map σ on the set of sequences on a 3-digit alphabet (ie. $\{(b_i)_{i=1}^{\infty} : b_i \in \{1,4,9\}\}$)

It is natural to extend the definition of iterated function system to higher dimensions:

<u>Defn</u> A finite collection $\varphi_1, \dots, \varphi_k$ of maps $\varphi_i : \mathbb{R}^d \to \mathbb{R}^d$ is called an <u>iterated function system</u> (on $\mathbb{R}^d$).

The associated map Φ acting on subsets of $\mathbb{R}^d$ is defined by:

$$\Phi(A) = \bigcup_{i=1}^{k} \varphi_i(A) .$$

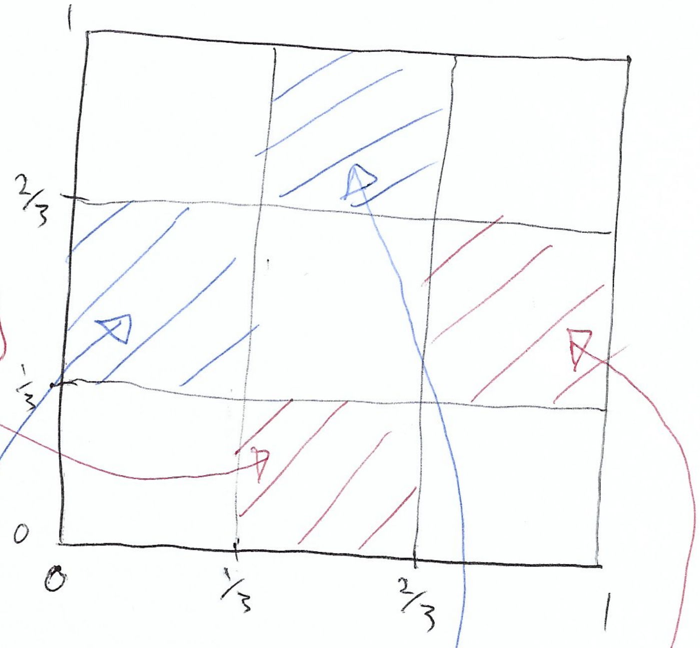
Example $d=2$

$$\varphi_1(x,y) := \left(\frac{x+1}{3}, \frac{y}{3}\right)$$

$$\varphi_1([0,1]^2) = [\tfrac{1}{3}, \tfrac{2}{3}] \times [0, \tfrac{1}{3}]$$

$$\varphi_2(x,y) := \left(\frac{x}{3}, \frac{y+1}{3}\right)$$

$$\varphi_2([0,1]^2) = [0, \tfrac{1}{3}] \times [\tfrac{1}{3}, \tfrac{2}{3}]$$



$$\varphi_3(x,y) := \left(\frac{x+2}{3}, \frac{y+1}{3}\right)$$

$$\varphi_3([0,1]^2) = [\tfrac{2}{3}, 1] \times [\tfrac{1}{3}, \tfrac{2}{3}]$$

$$\varphi_4(x,y) := \left(\frac{x+1}{3}, \frac{y+2}{3}\right)$$

$$\varphi_4([0,1]^2) = [\tfrac{1}{3}, \tfrac{2}{3}] \times [\tfrac{2}{3}, 1]$$

As usual, define $\Phi(A) = \bigcup_{i=1}^{4} \varphi_i(A)$

Let $F_0 = [0,1]^2$

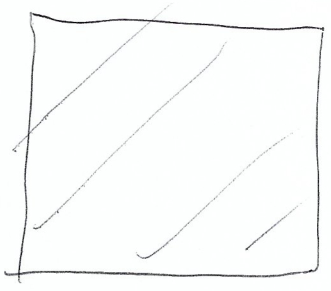
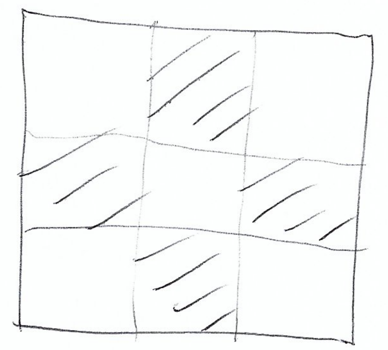
$F_1 = \Phi([0,1]^2) = \Phi(F_0)$

In general $F_n := \Phi^n([0,1]^2) = \Phi(F_{n-1})$

Let $F = \bigcap_{n=0}^{\infty} F_n = \bigcap_{n=0}^{\infty} \Phi^n([0,1]^2)$.

This is a Cantor set

Pictorially:

n	F_n
0	
1	
2