Week 10 (lecture 1)

What we'll cover

A] Inner products (examples)

B] Orthogonality

Recall the definition of inner product from Week 9.

We can define a norm starting from the inner prod.

<u>Theorem</u>: Each inner product space is a normed space

Proof we can define $\|\underline{v}\| = \sqrt{\langle v, v\rangle}$ and (a1) – (a3) are obvious. Let's check (a4). Let's define

$\underline{\tilde{v}} = \|\underline{v}\|^2 \underline{w} - \langle \underline{v}, \underline{w}\rangle \underline{v}$. We have

applying (b1)-(b3)

$$\|\underline{\tilde{v}}\|^2 = \langle \|\underline{v}\|^2 \underline{w} - \langle \underline{v}, \underline{w}\rangle \underline{v}, \|\underline{v}\|^2 \underline{w} - \langle \underline{v}, \underline{w}\rangle \underline{v}\rangle =$$

$$(\|\underline{v}\|^2)^2 \|\underline{w}\|^2 + \langle \underline{v}, \underline{w}\rangle^2 \|\underline{v}\|^2 - 2\|\underline{v}\|^2 \langle \underline{v}, \underline{w}\rangle^2$$

Property (a4) is trivially satisfied if $\underline{v} = 0$. So let us consider $\underline{v} \neq 0$, we get

Cauchy-Schwartz inequality

$$0 \leq \|\underline{v}\|^2 \left(\|\underline{v}\|^2 \|\underline{w}\|^2 - \langle v, w\rangle^2\right) \Rightarrow |\langle \underline{v}, \underline{w}\rangle| \leq \|\underline{v}\| \, \|\underline{w}\|$$

$$\Downarrow$$

$$\|\underline{v} + \underline{w}\|^2 = \langle \underline{v} + \underline{w}, \underline{v} + \underline{w}\rangle = \|\underline{v}\|^2 + \|\underline{w}\|^2 + 2\langle \underline{v}, \underline{w}\rangle \leq (\|\underline{v}\| + \|\underline{w}\|)^2 \Rightarrow (a4)$$

comment: while all inner product spaces are normed spaces the converse is not true. For instance there is no inner product in $\mathbb{R}^2$ that yields the norm of Ex 2] (Proof not given)

Ex 1] Euclidean product in $\mathbb{R}^n$

$\underline{y}, \underline{x} \in \mathbb{R}^n \qquad \langle \underline{x}, \underline{y} \rangle \equiv \underline{x} \cdot \underline{y} \equiv \underline{x}^T \underline{y} = \sum_{i=1}^{n} x_i^2$

Coursework: check that it satisfies (b1)-(b5)

Ex 1] Euclidean product in $\mathbb{R}^n$

Let $\underline{x}, \underline{y} \in \mathbb{R}^n$. The Euclidean inner product is

$$\underline{x} \cdot \underline{y} = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + \dots + x_n y_n = (x_1, \dots, x_n) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

The induced norm is $\|\underline{x}\| = \sqrt{x_1^2 + \dots + x_n^2}$.

Ex 2] Consider P_n (the vector space of real polynomial of degree at most n) and $p_1(t), p_2(t) \in P_n$. The following operation is an inner product

$$\langle p_1(t), p_2(t) \rangle = \int_0^1 p_1(t)\, p_2(t)\, dt$$

Ex 3] Consider the vector space $\mathbb{R}^{n \times n}$. The following operation is an inner product

$$\langle \underline{V}_1, \underline{V}_2 \rangle = \operatorname{Tr}(\underline{V}_1^T \underline{V}_2)$$ where

$\underline{V}_1, \underline{V}_2 \in \mathbb{R}^{n \times n}$ and Tr is called "trace" and indicates the sum of the diagonal elements of a square matrix

$$\operatorname{Tr}\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = a_{11} + a_{22} + \ldots + a_{nn}$$

Check the properties (b1) - (b5) listed at the end of week 9. Pay particular attention to (b4) - (b5). For instance $\operatorname{Tr}(\underline{V}_1^T \underline{V}_2)$ (without the transpose) does not satisfy these properties (as you can see in $\mathbb{R}^2$ by using $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$).

B] Definition: let V be a (real) vector space with an inner product $\langle , \rangle$. $\underline{V}_1, \underline{V}_2 \in V$ are orthogonal iff $\langle \underline{V}_1, \underline{V}_2 \rangle = 0$

Def: Let H be a vector subspace of V. $\underline{w} \in V$ is orthogonal to H if $\langle \underline{v}, \underline{w} \rangle = 0 \quad \forall v \in H$.

Def: The orthogonal complement of H (indicated by $H^\perp$) is the space of vectors $\underline{w} \in V$ such that $\langle \underline{w}, \underline{v} \rangle = 0 \quad \forall\, \underline{v} \in H$ (i.e. the space of vectors orthogonal to H).

Theorem: (i) $H^\perp$ is a vector subspace of V

(ii) If $H = \text{span}\{\underline{v}_1, \dots, \underline{v}_r\}$ then $\underline{w} \in H^\perp$ iff

$$\langle \underline{w}, \underline{v}_i \rangle = 0 \quad \forall\, i = 1, \dots, r$$

Proof: coursework

Ex 1] Let $H = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\} \subset \mathbb{R}^3$. Find $H^\perp$

$\underline{x} \in H^\perp$ iff $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Thus we have to solve the linear system

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \end{array} \right) \overset{REF}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right) \rightsquigarrow \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Thus $H^\perp = \text{span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$

Theorem: let $A \in \mathbb{R}^{m \times n}$ then

(i) $N(A) = \text{col}(A^T)^\perp$

(iii) $N(A^T) = \text{col}(A)^\perp$

Proof. $\underline{x} \in \mathbb{R}^n$ belongs to $N(A)$ iff $A\underline{x} = 0 \Rightarrow$

x is orthogonal to every row of A.

The rows of A are the columns of A^T.

So $\underline{x}$ is orthogonal to every column of A^T.

The column space of A^T ($\text{col}(A^T)$) is thus

orthogonal to $\underline{x} \Rightarrow \underline{x} \in (\text{col}(A^T))^\perp$

(ii) Same as before with $A \leftrightarrow A^T$

Ex 2] $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}$ $A^T = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{pmatrix}$

$\text{col}(A^T) = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$. Thus $N(A) = \text{col}(A^T)^\perp$

is the vector subspace of $\underline{x} \in \mathbb{R}^3$ such that

$\underline{x} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 0$ $\quad (x_1 \ x_2 \ x_3) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = x_1 + 2x_2 + 3x_3 = 0$

Thus $N(A) = \text{col}(A^T)^\perp = \text{span}\left\{ \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right\}$

Geometrically it is a plane in $\mathbb{R}^3$ orthogonal

to the line $\text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$

What we'll cover (lectures 2-3)

A] Orthogonal and Orthonormal sets

B] Orthogonal matrices

C] Gram-Schmidt orthonormalisation

A] Definition: a set $S = \{ \underline{v}_1, \dots, \underline{v}_n \}$ is orthogonal if its vectors are pairwise orthogonal, i.e. $\langle \underline{v}_i, \underline{v}_j \rangle = 0$ whenever $i \neq j$

Ex 1] consider the following $\underline{v}_i$'s in $\mathbb{R}^3$

$$\underline{v}_1 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} ; \quad \underline{v}_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} , \quad \underline{v}_3 = \begin{pmatrix} -1 \\ -4 \\ 7 \end{pmatrix}$$

We have $\langle \underline{v}_1, \underline{v}_2 \rangle = \underline{v}_1 \cdot \underline{v}_2 = (3 \; 1 \; 1) \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\langle \underline{v}_1, \underline{v}_3 \rangle = \underline{v}_1 \cdot \underline{v}_3 = (3 \; 1 \; 1) \begin{pmatrix} -1 \\ -4 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\langle \underline{v}_2, \underline{v}_3 \rangle = \underline{v}_2 \cdot \underline{v}_3 = (-1 \; 2 \; 1) \begin{pmatrix} -1 \\ -4 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Theorem: $S = \{ \underline{v}_1, \dots \underline{v}_r \}$ is an orthogonal set with $\underline{v}_i \neq \underline{0} \; \forall i$. Then S is linearly independent

Proof: we start from

$$\alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 + \dots + \alpha_r \underline{v}_r = \underline{0} \quad \text{with } \alpha_i \in \mathbb{R}$$

and take the scalar product of this relation with $\underline{v}_i$

$$\langle \underline{v}_i , \sum_{j=1}^{r} \alpha_j \underline{v}_j \rangle = \langle \underline{0}, \sum_{j=1}^{r} \alpha_j \underline{v}_j \rangle = 0$$

$$\parallel$$

$$\sum_{j=1}^{r} \alpha_j \langle \underline{v}_i , \underline{v}_j \rangle = \alpha_i \|\underline{v}_i\|^2$$

Since $\underline{v}_i \neq 0$, then $\|\underline{v}_i\|^2 > 0$ and $\alpha_i = 0 \quad \forall i$.

Definition: An orthogonal basis is a basis which is also an orthogonal set

Theorem: Let $B = \{\underline{v}_1, \dots, \underline{v}_n\}$ be an orthogonal basis for a real vector space. The coordinates of $\underline{v} \in V$ according to B are

$$[\underline{v}]_B = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \quad \text{with} \quad \alpha_i = \frac{\langle \underline{v}_i , \underline{v} \rangle}{\|\underline{v}_i\|^2} \quad i = 1, \dots, n$$

This means $\underline{v} = \sum_{j=1}^{n} \alpha_j \underline{v}_j$. Again take the scalar product of this relation with $\underline{v}_i$

$$\langle \underline{v}_i, \underline{v} \rangle = \langle \underline{v}_i, \sum_{j=1}^{n} \alpha_j \underline{v}_j \rangle = \sum_{j=1}^{n} \alpha_j \langle \underline{v}_i, \underline{v}_j \rangle \underset{\text{orth. set}}{=}$$

$$\alpha_i \langle \underline{v}_i, \underline{v}_i \rangle = \alpha_i \| \underline{v}_i \|^2 \Rightarrow \alpha_i = \frac{\langle \underline{v}_i, \underline{v} \rangle}{\| \underline{v}_i \|^2}$$

Ex 2] By using the basis of Ex 1] above we can calculate $[\underline{v}]_B$ for $\underline{v} = \begin{pmatrix} 6 \\ 1 \\ -8 \end{pmatrix}$ by using the theorem just proved. We have

$\| \underline{v}_1 \|^2 = 9 + 1 + 1 = 11$; $\| \underline{v}_2 \|^2 = 1 + 4 + 1 = 6$ and

$\| \underline{v}_3 \|^2 = 1 + 16 + 49 = 66$. Thus $[\underline{v}]_B = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$ with

$$\alpha_1 = \frac{\underline{v}_1 \cdot \underline{v}}{\| \underline{v}_1 \|^2} = \frac{1}{11} (6 \ 1 \ -8) \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \frac{11}{11} = 1$$

Check it by calculating $\sum_{i=1}^{3} \alpha_i \underline{v}_i$

$$\alpha_2 = \frac{\underline{v}_2 \cdot \underline{v}}{\| \underline{v}_2 \|^2} = \frac{1}{6} (6 \ 1 \ -8) \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = -2$$

$$\alpha_3 = \frac{\underline{v}_3 \ \underline{v}}{\| \underline{v}_3 \|^2} = \frac{1}{66} (6 \ 1 \ -8) \begin{pmatrix} -1 \\ -4 \\ 7 \end{pmatrix} = -1$$

Definition: An orthonormal set is an orthogonal set consisting of unit vectors, i.e. $S = \{ \underline{v}_1, \ldots, \underline{v}_n \}$ with

$\forall i, j \quad \langle \underline{v}_i, \underline{v}_j \rangle = \delta_{ij}$ ⟵ Kronecker delta

Ex 3] Consider again $\mathbb{R}^3$ and the orthogonal basis B of the previous two examples. We can construct an orthonormal basis as follows

$$\underline{u}_1 = \frac{\underline{v}_1}{\|\underline{v}_1\|} = \frac{1}{\sqrt{11}}\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \quad \underline{u}_2 = \frac{\underline{v}_2}{\|\underline{v}_2\|} = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

$$\underline{u}_3 = \frac{\underline{v}_3}{\|\underline{v}_3\|} = \frac{1}{\sqrt{66}}\begin{pmatrix} -1 \\ -4 \\ 7 \end{pmatrix}$$

Comment: Orthonormal basis are of course not unique.

Consider the standard basis in $\mathbb{R}^3$

$$\underline{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad \underline{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \underline{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ and}$$

check that it is orthonormal

B] Consider the change of basis from the standard basis $\underline{X} = \{\underline{x}_1, \underline{x}_2, \underline{x}_3\}$ in $\mathbb{R}^3$ to the basis $U = \{\underline{u}_1, \underline{u}_2, \underline{u}_3\}$ above. We have

$$[id]_X^U = \begin{pmatrix} 3/\sqrt{11} & -1/\sqrt{6} & -1/\sqrt{66} \\ 1/\sqrt{11} & 2/\sqrt{6} & -4/\sqrt{66} \\ 1/\sqrt{11} & 1/\sqrt{6} & 7/\sqrt{66} \end{pmatrix} = Q$$

Check that $Q^T Q = Q Q^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Definition: A matrix $Q \in \mathbb{R}^{n \times n}$ is orthogonal iff $Q^T Q = I$

Theorem: Let $Q \in \mathbb{R}^{n \times n}$ be orthogonal

(i) Q is invertible with $Q^{-1} = Q^T$ (thus we also have $Q Q^T = I$)

(ii) If $\{\underline{v}_1, \dots, \underline{v}_n\}$ is an orthonormal set in $\mathbb{R}^n$ then $\{Q \underline{v}_1, \dots, Q \underline{v}_n\}$ is another orthonormal set in $\mathbb{R}^n$

Proof: Coursework

Comment: You can consider an orthonormal set of n vectors $\{\underline{v}_1, \dots, \underline{v}_n\}$ in $\mathbb{R}^m$. Then $m \geq n$ (Proof: homework). If $m > n$, then the matrix A whose columns are the $\underline{v}_i$'s is a rectangular matrix satisfying $A^T A = I \leftarrow$ ($n \times n$ identity matrix).

Sketch of the proof: Let $(\underline{v}_i)_j$ be the j^{th} component of $\underline{v}_i \in \mathbb{R}^m$. We have

$$A^T = \begin{pmatrix} (v_1)_1 & \cdots & (v_1)_m \\ (v_2)_1 & \cdots & (v_2)_m \\ \vdots & & \\ (v_n)_1 & \cdots & (v_n)_m \end{pmatrix}$$

Thus $\left(A^T A\right)_{ij} = \langle \underline{v}_i, \underline{v}_j \rangle = \delta_{ij}$

From what we saw so far about orthogonal matrices we have

Corollary: let $A \in \mathbb{R}^{m \times n}$ be a matrix with orthonormal columns. Then for any $\underline{x}, \underline{y} \in \mathbb{R}^n$ we have

$$(A\underline{x}) \cdot (A\underline{y}) = \underline{x} \cdot \underline{y}$$

This implies $\|A x\| = \|x\|$ and $(A\underline{x}).(A\underline{y}) = 0$ iff $\underline{x} \cdot \underline{y} = 0$. Proof: home work.

c] Gram-Schmidt process

Consider a vector space V and a basis $\{\underline{v}_1, \dots, \underline{v}_n\}$ for V. We can construct an orthogonal basis $\{\underline{w}_1, \dots, \underline{w}_n\}$ and an orthonormal one $\{\underline{u}_1, \dots, \underline{u}_n\}$ as follows

- $\underline{w}_1 = \underline{v}_1$ and $\underline{u}_1 = \underline{w}_1 / \|\underline{w}_1\|$ ($\|w_1\| \neq 0$ since $\underline{w}_1 \neq 0$)

- $$\underline{w}_2 = \underline{v}_2 - \frac{\langle \underline{w}_1, \underline{v}_2 \rangle}{\langle \underline{w}_1, \underline{w}_1 \rangle} \underline{w}_1 = \underline{v}_2 - \langle \underline{u}_1, \underline{v}_2 \rangle \underline{u}_1; \quad \underline{u}_2 = \frac{\underline{w}_2}{\|\underline{w}_2\|}$$

$\vdots$

- $$\underline{w}_n = \underline{v}_n - \sum_{i=1}^{n-1} \frac{\langle \underline{w}_i, \underline{v}_n \rangle}{\langle \underline{w}_i, \underline{w}_i \rangle} \underline{w}_i = \underline{v}_n - \sum_{i=1}^{n-1} \langle \underline{u}_i, \underline{v}_n \rangle \underline{u}_i; \quad \underline{u}_n = \frac{\underline{w}_n}{\|\underline{w}_n\|}$$

Proof: Let us calculate

$$\langle \underline{w}_1, \underline{w}_2 \rangle = \langle \underline{v}_1, \underline{v}_2 - \frac{\langle \underline{v}_1, \underline{v}_2 \rangle}{\|\underline{w}_1\|^2} \underline{v}_1 \rangle = \langle \underline{v}_1, \underline{v}_2 \rangle - \langle \underline{v}_1, \underline{v}_2 \rangle = 0$$

Complete the proof by induction: assume that $\{\underline{w}_1, \dots, \underline{w}_{n-1}\}$ are orthogonal (and so $\{\underline{u}_1, \dots, \underline{u}_{n-1}\}$ are orthonormal), then

$$\langle \underline{w}_n, \underline{w}_i \rangle = \langle \underline{v}_n - \sum_{j=1}^{n-1} \frac{\langle \underline{w}_j, \underline{v}_n \rangle}{\langle \underline{w}_j, \underline{w}_j \rangle} \underline{w}_j, \underline{w}_i \rangle =$$

$$\langle \underline{v}_n, \underline{w}_i \rangle - \sum_{j=1}^{n-1} \frac{\langle \underline{w}_j, \underline{w}_i \rangle}{\langle \underline{w}_j, \underline{w}_j \rangle} \langle \underline{v}_n, \underline{w}_i \rangle = 0$$

zero unless $i = j$ since $\{\underline{w}_1, \dots, \underline{w}_{n-1}\}$ is orthonor.

Then $\{\underline{w}_1, \dots, \underline{w}_n\}$ are orthogonal and $\{\underline{u}_1, \dots, \underline{u}_n\}$ are orthonormal ✓

Corollary: If H is a vector subspace of V and $\text{span}\{\underline{v}_1, \dots, \underline{v}_n\} = H$. The Gram-Schmidt process provide an orthonormal basis for H.

Ex 4] Consider $\mathbb{R}^4$ and

$$\underline{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \underline{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \quad \underline{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 6 \end{pmatrix}$$

- $\underline{w}_1 = \underline{x}_1$ and $\underline{u}_1 = \dfrac{1}{\|\underline{x}_1\|}\underline{x}_1 = \dfrac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

- $\underline{w}_2 = \underline{x}_2 - \langle \underline{u}_1, \underline{x}_2 \rangle \underline{u}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix} - \dfrac{1}{2}\left((1\ 1\ 1\ 1) \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix} \right) \dfrac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} =$

$$= \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \underline{u}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

- $\underline{w}_3 = \underline{x}_3 - \langle \underline{u}_1, \underline{x}_3 \rangle \underline{u}_1 - \langle \underline{u}_2, \underline{x}_3 \rangle \underline{u}_2 =$

$$\begin{pmatrix} 0 \\ 0 \\ 2 \\ 6 \end{pmatrix} - \frac{1}{2}\left((1\ 1\ 1\ 1)\begin{pmatrix} 0 \\ 0 \\ 2 \\ 6 \end{pmatrix} \right) \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{2}}\left((-1\ 0\ 0\ 1)\begin{pmatrix} 0 \\ 0 \\ 2 \\ 6 \end{pmatrix} \right) \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} =$$

$$\begin{pmatrix} 0 \\ 0 \\ 2 \\ 6 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - 3\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

Normalise $\underline{w}_i$ to obtain the orthonormal basis $\{\underline{u}_i\}$