Main Examination period 2023 – January – Semester A

MTH5130: Number Theory

Duration: 2 hours

The exam is intended to be completed within **2 hours**. However, you will have a period of **4 hours** to complete the exam and submit your solutions.

You should attempt ALL questions. Marks available are shown next to the questions.

All work should be **handwritten** and should **include your student number**. Only one attempt is allowed – **once you have submitted your work, it is final**.

In completing this assessment:

- You may use books and notes.
- You may use calculators and computers, but you must show your working for any calculations you do.
- You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.
- You must not seek or obtain help from anyone else.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;

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Question 1 [15 marks]. Find an integer solution to the equation

$$35x + 55y + 77z = 1.$$

in x, y, z. Show your working. [**15**]

Question 2 [16 marks]. Are the following assertions true or false? Explain your answers.

(a) 7 is a primitive root mod 11. [**4**]

(b) 25 is a quadratic residue mod 11. [**4**]

(c) 2 is a square mod 9. [**4**]

(d) 17^{253} is a solution to the congruence equation $x^2 \equiv -1 \bmod 1013$. You may assume 1013 is a prime number. [**4**]

Question 3 [24 marks].

(a) Compute the continued fraction $[1; 1, \overline{1, 2}]$. Show your working. [**8**]

(b) Find an example of a good rational approximation to

$$e = 2.71828... = [2; 1, 2, 1, 1, 4, \ldots].$$

Justify your answer. [**6**]

(c) Prove that if an irrational number has a purely periodic continued fraction then it is a quadratic irrational. State clearly any result you are using from lectures. [**10**]

Question 4 [20 marks].

(a) Compute the continued fraction of $\sqrt{23}$. Show your working. [**8**]

(b) Find the fundamental solution to $x^2 - 23y^2 = \pm 1$. Justify your answer. In doing so, state clearly any results you are using from lectures. [**4**]

(c) Using (b), compute the 7-th convergent to $\sqrt{23}$. [**8**]

Question 5 [10 marks]. Find an integer solution to

$$x^2 + y^2 = 725$$

using Hermite's algorithm. [**10**]

Question 6 [15 marks]. Determine which of the following real numbers are algebraic integers. State clearly any results you are using from lectures and justify your answers.

(a) $\dfrac{26}{3}$. [**3**]

(b) $\pi = 3.1415926535....$ [**3**]

(c) $1 + \dfrac{\sqrt{21}}{2}$. [**3**]

(d) $\dfrac{-1 + \sqrt{-3}}{2}$. [**3**]

(e) $1 + \sqrt[3]{3}$. [**3**]

End of Paper.