EXTRA TUESDAY TUTORIALS WEEKS 10-11-12

Essentials lecture 1.

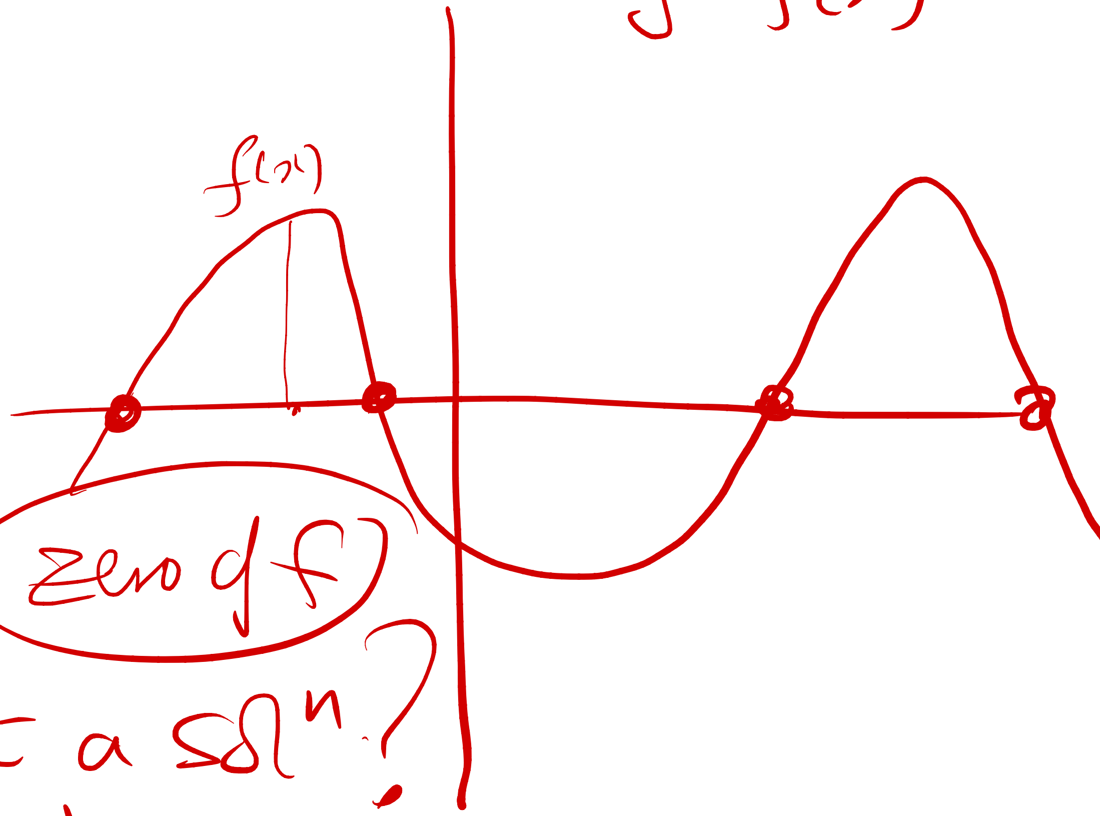
Chapter 1 overview.

$\dot{x} = f(x)$ ~~t~~ , $x \in \mathbb{R}$ autonomous.

$\ddot{x} + x = t\sin t$

$f(x)$ – graph.

$y = f(x)$



$\dot{x} = f(x)$

$f(x) = 0$ for $x = x^*$ zero of f

$x(t) = x^*$ – is it a soln? $\forall t$

$x(t) = x^*$, $\frac{dx}{dt} = 0$, fixed point

$f(x^*) = 0$

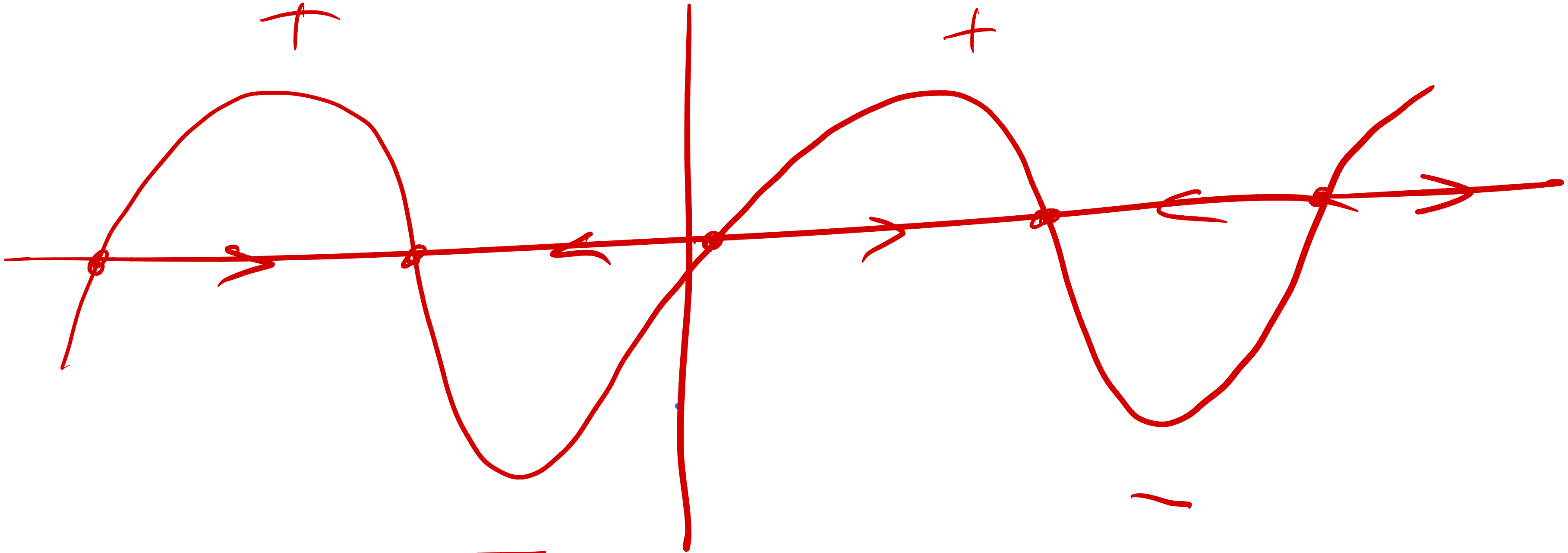
$\frac{dx^*(t)}{dt} = f(x^*) -$

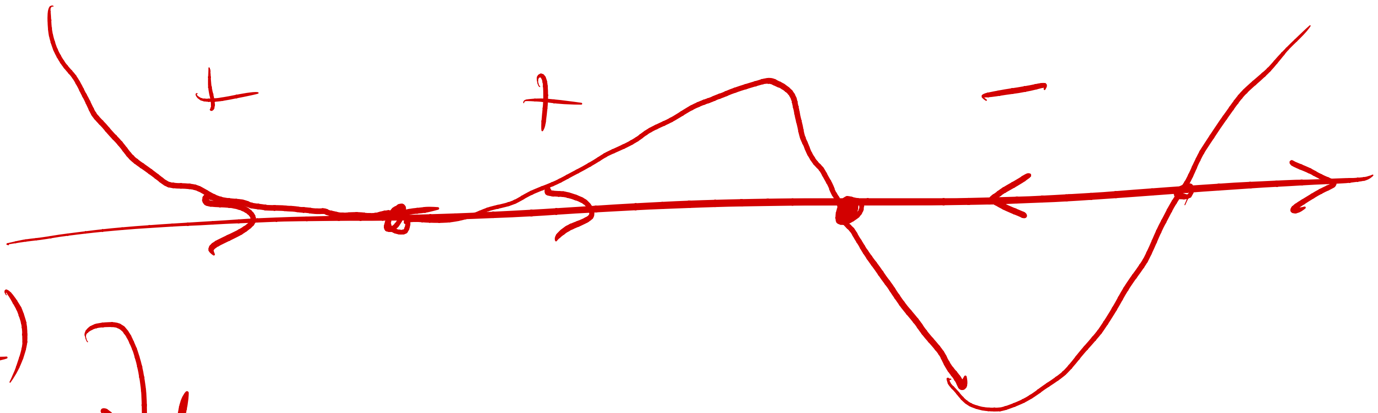
zeros of f = fixed of $\dot{x} = f(x)$

$f(x) \neq 0$ at the non-fixed point.

$\dot{x} > 0$

< 0

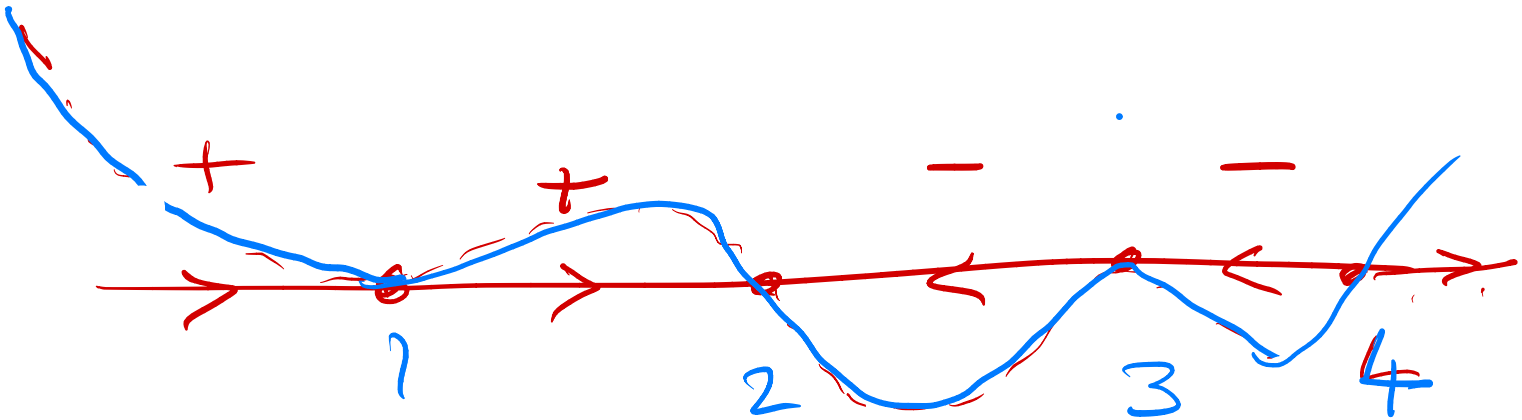




$\dot{x} = f(x)$

phase portraits

$\dot{x} = f(x)$



x^{2n}

$$\dot{x} = (x-1)^2 (x-2)^1 (x-3)^2 (x-4)^1$$

$$(2-x)$$

$$+x^6$$

Real line

$x(t) = e^t$ $\quad$ $\dot{x} = \frac{dx}{dt} = e^t = x$

Soln?

$\dot{x} = x$

$x(t) = \sin t$, $\quad \dot{x} = \cos t \quad = f(x)$

$\dot{x} = \cos t = \sqrt{1 - \sin^2 t} = \sqrt{1 - x^2}$. ✓

$t \in (0, \pi)$ – valid differential.

Circle

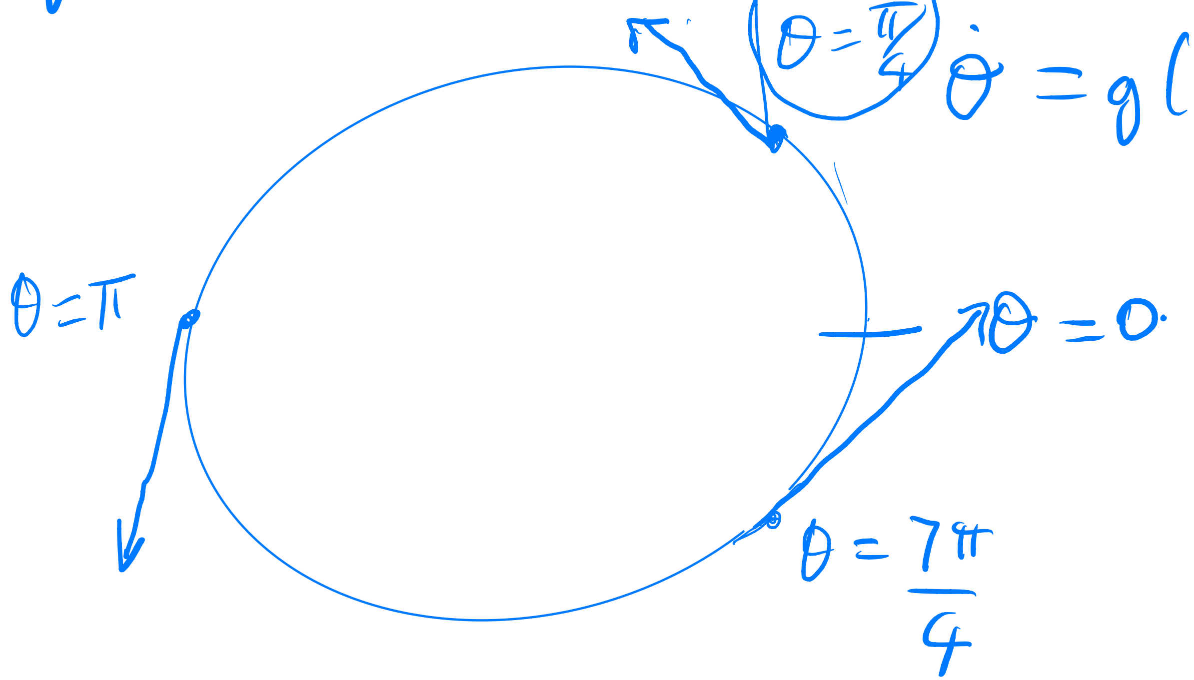
$\theta(t) = \sin(t)$, $\quad \dot{\theta} = \cos t = +\sqrt{1 - \sin^2 t}$

$\dot{\theta} = \sqrt{1 - \theta^2}$?

$\dot{\theta} = g(\theta)$ — ODE on the circle S^1

DS

$g(\theta) = \sqrt{1-\theta^2}$



$\sqrt{1-\theta^2}$

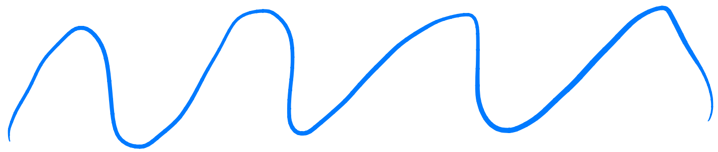
$g(\theta) = \sqrt{1-\theta^2}$ ✗ Not periodic in θ.

$$g(\theta) = \sum_{k=-\infty}^{+\infty} a_k \cos 2k\theta + b_k \sin 2k\theta.$$

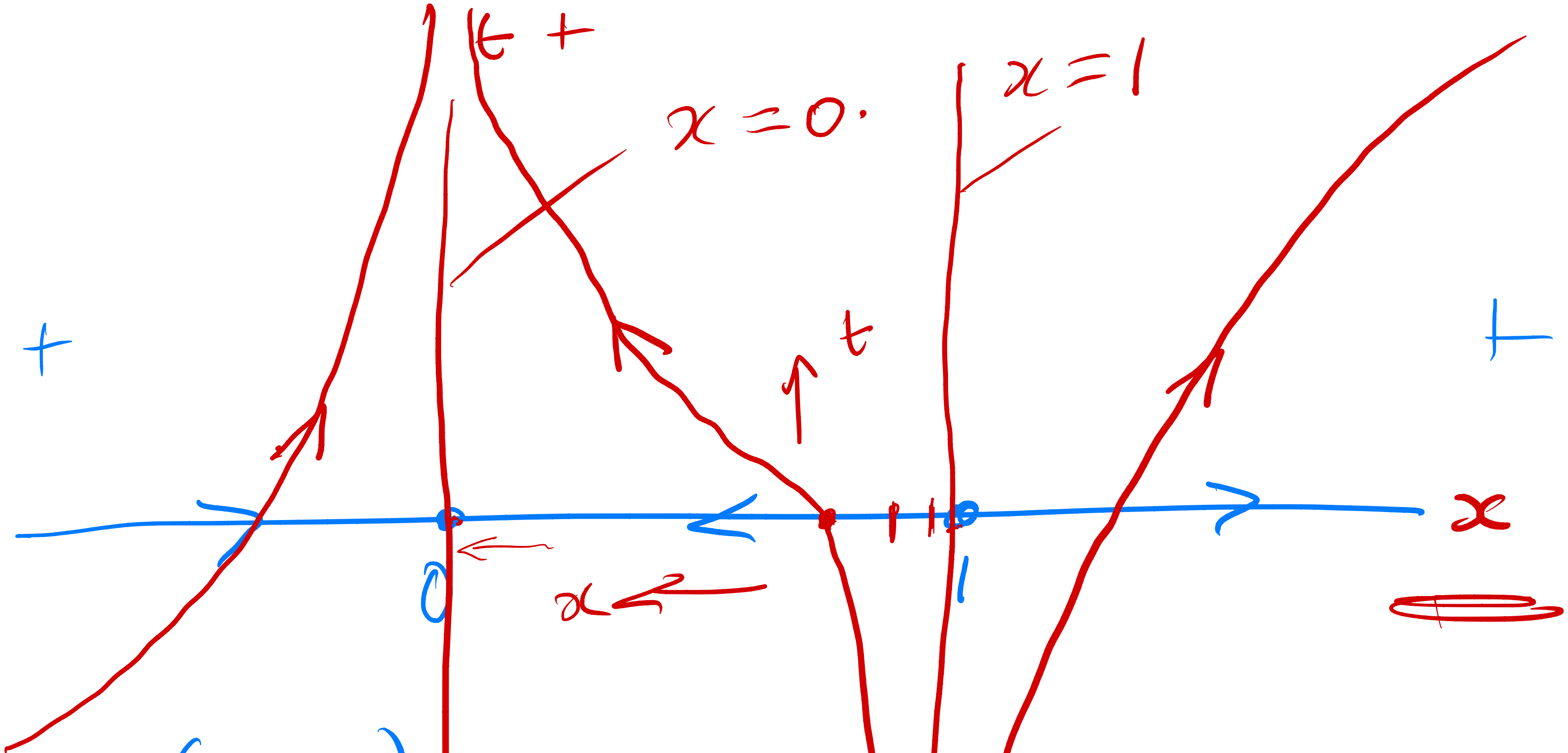
$$g(\theta) = 1 - \cos\theta$$

$$1 - \cos 2\theta$$

$$\sin\theta - \cos\theta.$$



$$g(\theta) = \sinh\theta.$$



$$\dot{x} = x(x-1)$$

$$\int \frac{dx}{x(x-1)} = \int dt$$

$$\int \frac{A}{x} + \frac{B}{x-1}\, dx = \int dt$$

x-t soln curves.

phase portrait.

Chk p6

$\dot{x} = f(x)$

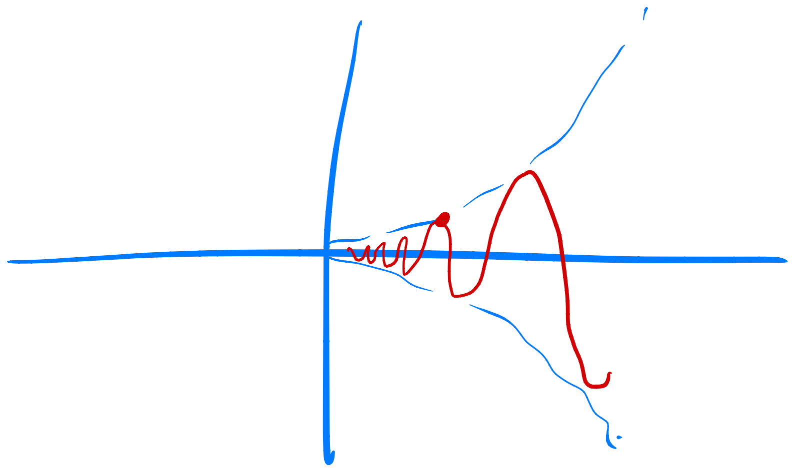
zero $x = x^*$

How would you hope to find its stability.

Linear stability

$f'(x)$

$x^2 \sin\left(\frac{1}{x}\right)$



eval at $x = x^*$.

$f'(x^*) > 0$ unstable

$f'(x^*) < 0$ stable.

$f'(x^*) = 0$. ??

Chapter 4

$$\underline{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\dot{\underline{x}} = \underline{A}\,\underline{x} \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$P^{-1} A P = J$$

$$\underline{z} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \underline{w} = \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\dot{\underline{z}} = A\,\underline{z} \qquad \underline{z} = P\underline{w} \qquad \underline{w} = P^{-1}\underline{z}$$

$$\dot{\underline{z}} = P\dot{\underline{w}} = \underline{A}\,\underline{P}\,\underline{w} \implies \dot{\underline{w}} = \underline{\underline{P^{-1}}}\,\underline{A}P\underline{w}$$

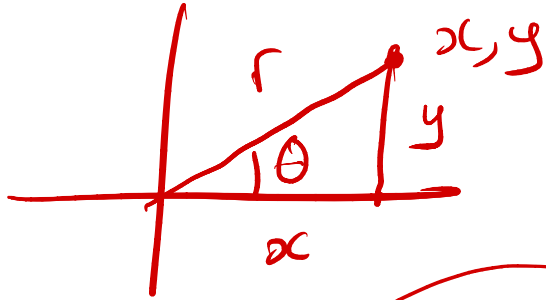
Choose P s.t. $P^{-1}AP = J$

$$J = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

polar coordinates

$r^2 = x^2 + y^2$, $\tan\theta = \dfrac{y}{x}$

" α β one "



$\dot{r} = \alpha r$ $\qquad$ $\dot{\theta} = \beta$

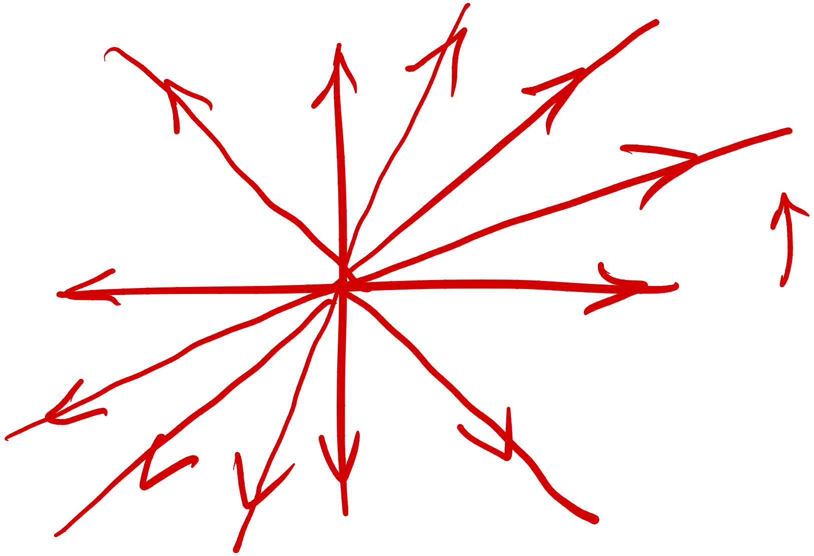
radial motion $\qquad$ angular.

$\alpha > 0$ $r \nearrow$

$\alpha < 0$ $r \swarrow$

$\beta = 0$

$\alpha > 0$

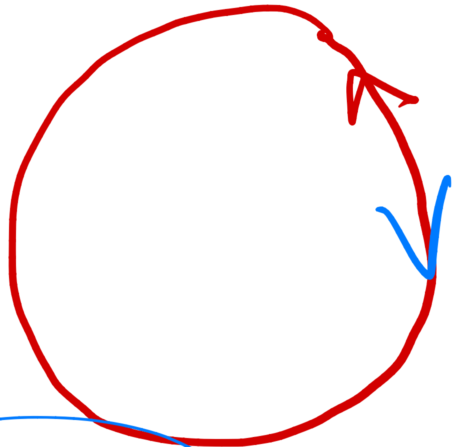


$\uparrow \quad \dot{\theta} = \beta > 0$

$\beta = 1$

$\alpha = 0$

$\dot{r} = 0$



$\beta = -1$

$\alpha = 0$

$\dot{r} < 0$

$\alpha < 0.$

$\beta = 1$

$\dot{r} = \alpha r$

$-\dot{\rho} = \alpha[-\rho]$

$$J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

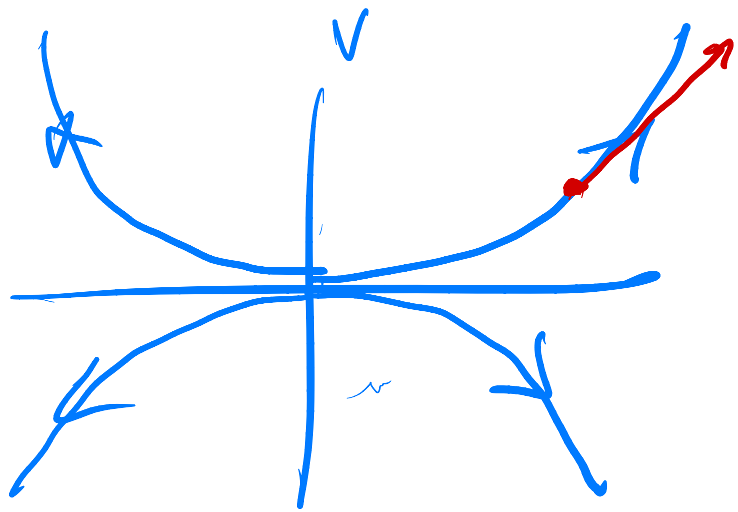
$$\dot{u} = \lambda_1 u$$
$$\dot{v} = \lambda_2 v$$

$$\frac{du}{\lambda_1 u} = \frac{dv}{\lambda_2 v}.$$

$$v = C u^{\frac{\lambda_2}{\lambda_1}}$$

$$\dot{u} = 2u$$
$$\dot{v} = 3v$$

$$\left.\begin{array}{l} \lambda_1 = 2 \\ \lambda_2 = 3 \end{array}\right\}$$



$$v = C u^{3/2}.$$

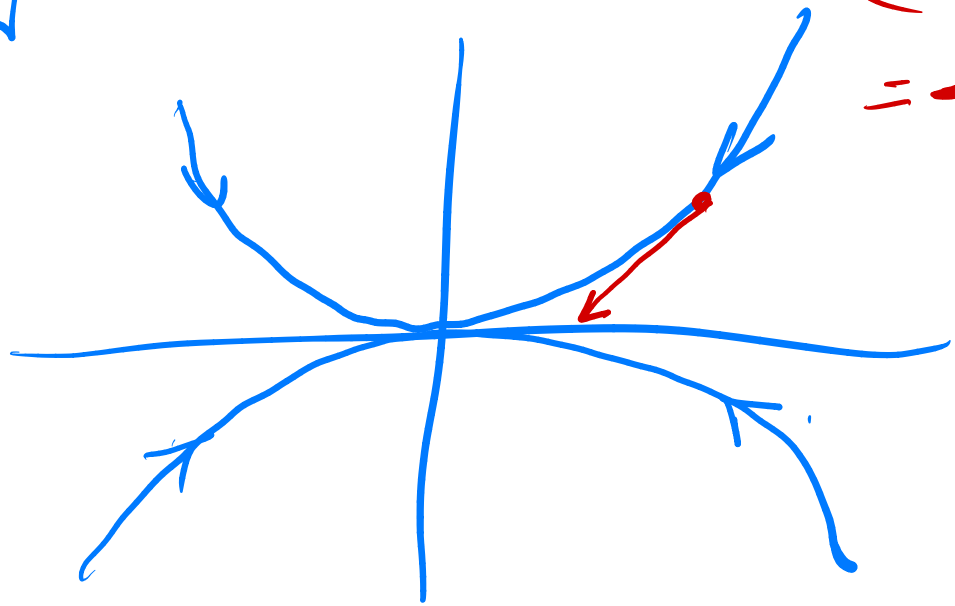
$$\frac{dv}{du} = \frac{3}{2} C u^{1/2}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \qquad \lambda_1 = \lambda_2 = 2 \quad \text{unstable improper}$$

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\left.\begin{aligned} \dot{u} &= -2u \\ \dot{v} &= -3v \end{aligned}\right\}$$

reverse vector field $(-2u,-3v) = -(2u,3v)$



$$\dot{\underline{x}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \underline{x}$$

$$\dot{x} = x + y$$
$$\dot{y} = y$$

improper node

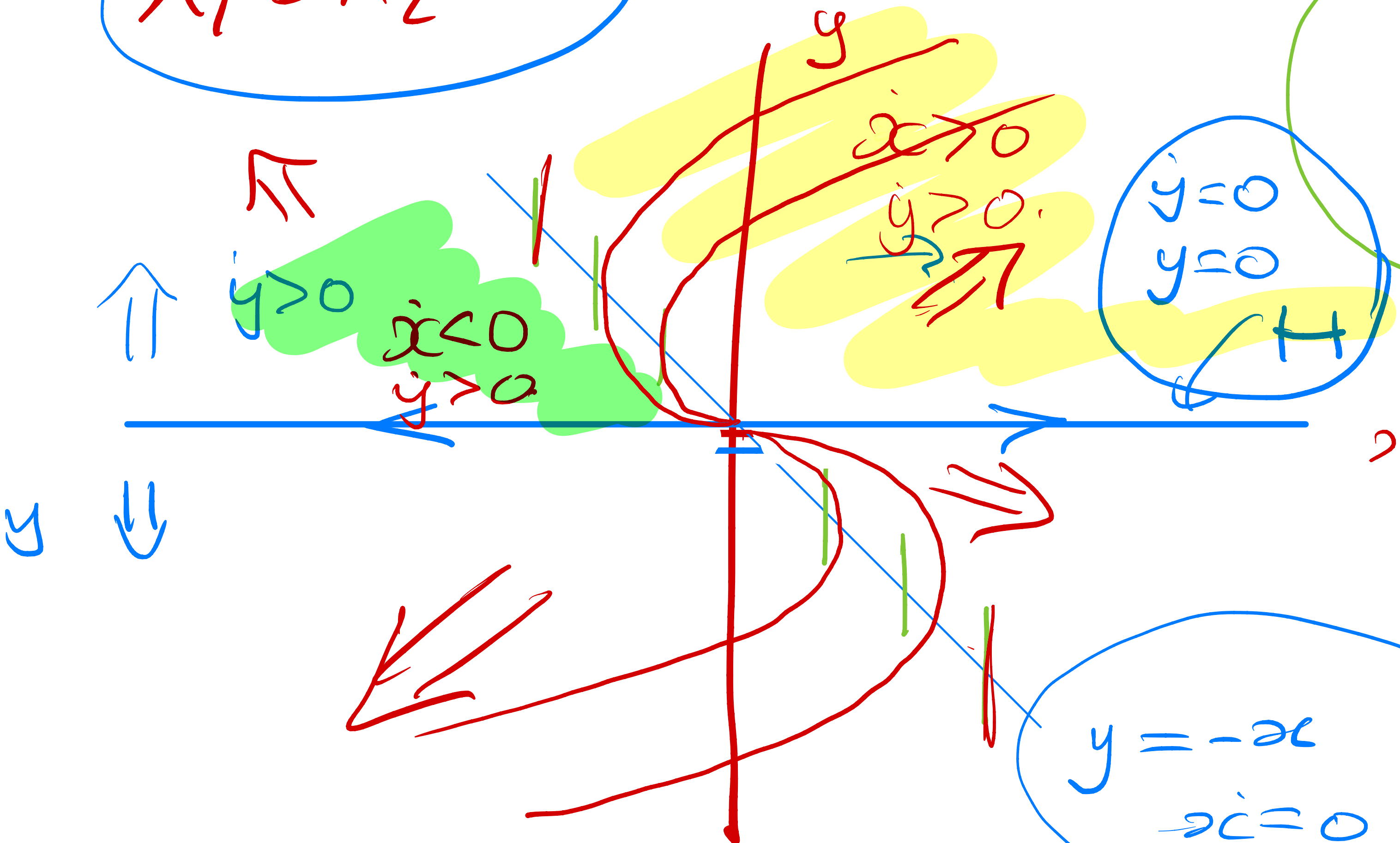
$\lambda_1 = \lambda_2 = 1$

unstable

nullclines

$\dot{x} = 0$
$x + y = 0$

$\dot{y} = 0$
$y = 0$
x-axis



$y = -x$
$\dot{x} = 0$

Question 1 [28 marks]. One dimensional systems on $\mathbb{R}$ and $\mathbb{S}$

(a) Consider the following dynamical systems on the line $\mathbb{R}$, given by the ordinary differential equations

(i) $\dot{x} = x^2(x^6 + x^3 - 1)$,

(ii) $\dot{x} = \exp(-x) - \tanh(x)$,

(iii) $\dot{x} = \sin(x) - \tanh(x)$.

Investigate each system, and deduce the phase portrait on $\mathbb{R}$ for each system. [12]

(b) Sketch the phase portrait of a flow on the circle, $\mathbb{S}$, which has exactly 4 fixed points: one being linearly stable; one being linearly unstable; and two being saddle-nodes. Identify the basin of attraction of each of the fixed points in the phase portrait diagram. [6]

(c) Find a differentiable function $f : \mathbb{S} \to \mathbb{R}$ such that the differential equation $\dot{\theta} = f(\theta)$ generates the phase portrait described in (b). [6]

(d) Explain in words the likely structure of a phase portrait given by $\dot{\theta} = f^2(\theta)$, for **any** differentiable function $f : \mathbb{S} \to \mathbb{R}$. [4]

Q1 2021

$$\dot{x} = x^2\left(x^6 + x^3 - 1\right) = f(x) \approx x^8 \quad \text{large } x$$

FPs $\quad x^2(x^6 + x^3 - 1) = 0 \qquad x = 0$

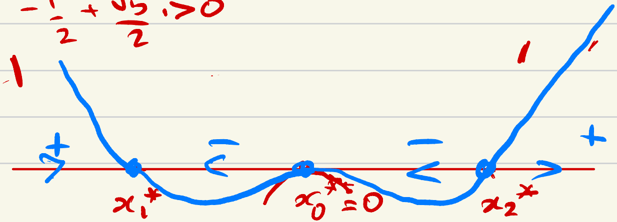
$$x^6 + x^3 - 1 = 0$$

$$\underset{a}{(x^3)^2} + \underset{b}{x^3} \underset{c}{-1} = 0 \qquad x^3 = \frac{-1 \pm \sqrt{1 - 4(-1)}}{2}$$

$$= -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$$

$$\left.\begin{aligned} x_1^3 &= -\frac{1}{2} - \frac{\sqrt{5}}{2} < 0 \\ x_2^3 &= -\frac{1}{2} + \frac{\sqrt{5}}{2} > 0 \end{aligned}\right\} \; x_1^* < 0, \; x_2^* > 0$$

$\dot{x} \approx x^8$ as $|x| \to \infty$



(ii) $\dot{x} = \underline{\exp(-x)} - \underline{\tanh(x)}$

$y = \exp(-x)$
$y = \tanh(x)$ } intersections

$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x > 0$

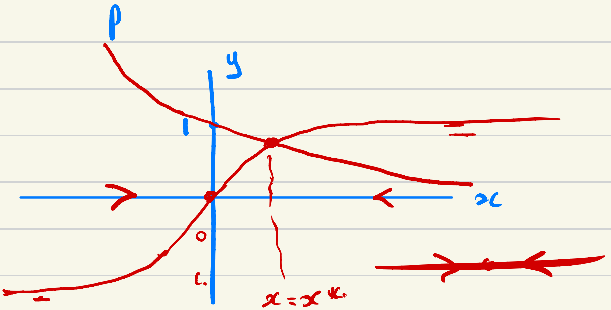
$e^{-x} \to 0$ as $x \to \infty$

$e^{-x} \to \infty$ as $x \to -\infty$

$\tanh(x)$
$=$
$\frac{\sinh(x)}{\cosh(x)} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$

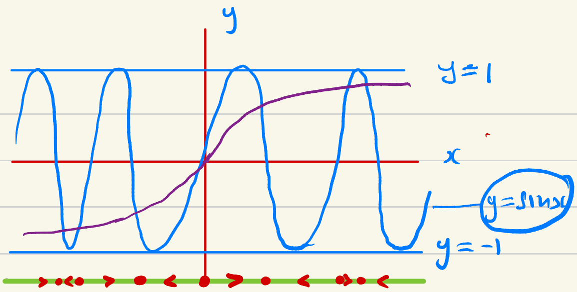
$\lim_{x \to \infty} \to 1^{-}$

$\lim_{x \to -\infty} \to -1^{+}$



$\dot{x} = \sin x - \tanh(x)$

Plot $y = \sin x$ & $y = \tanh(x)$ and look for intersections to locate fixed points



$\tanh(x)$ asym to $y = 1^-$ as $x \to \infty$

$y = -1^+$ as $x \to -\infty$

$\sin x$ attains max and min $y = \pm 1$ at $x = 2n\pi$, $n \in \mathbb{Z}$.

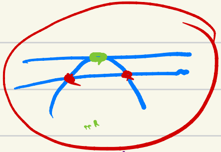
Note for $x \approx 0$

$\sin x - \tanh x \sim \frac{x^3}{6}$

$\therefore$ +ve $x > 0$

-ve $x < 0$

so $x = 0$ is unstable FP



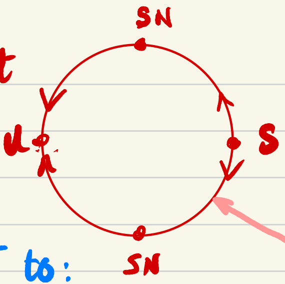
graphs of $\sin x$ and $\tanh x$ switch position at remaining fixed point

For small x,

$$\sin x - \frac{\sinh(x)}{\cosh(x)} = x - \frac{x^3}{6} + \dots - \frac{\left(x + \frac{x^3}{6} + \dots\right)}{\left(1 + \frac{x^2}{2} + \dots\right)} = x - \frac{x^3}{6} - \left(x + \frac{x^3}{6}\right)\left(1 - \frac{x^2}{2} + \right)$$

$$= x - \frac{x^3}{6} - x - \frac{x^3}{6} + \frac{x^3}{2} = \frac{x^3}{3} \dots$$

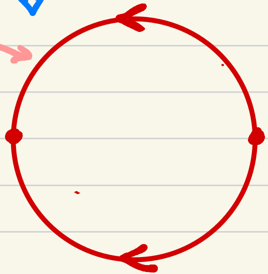
What we want

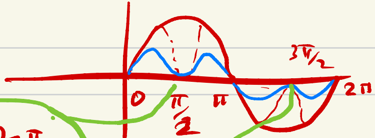


Need to add FPs at $x = \pi/2, \frac{3\pi}{2}$ to:

$\Downarrow$

$\boxed{\dot\theta = \sin\theta}$





FP at $\theta = \pi/2$

FP at $\theta = 3\pi/2$

$$\boxed{\dot\theta = \sin\theta\left(1-\cos\left(\theta-\tfrac{\pi}{2}\right)\right)\left(1-\cos\left(\theta-\tfrac{3\pi}{2}\right)\right)} = f(\theta)$$

so the factor $1-\cos(\theta-\pi/2)$ is >0 except for 0 at $\theta = \pi/2$

i.e.

$$\boxed{\begin{array}{c}\operatorname{sgn}(f(\theta)) = \operatorname{sgn}(\sin\theta) \\ \text{except for } \theta = \pi/2,\ \theta = 3\pi/2\end{array}}$$

$\boxed{\dot{\theta} = f(\theta)}$ and $\boxed{\dot{\theta} = (f(\theta))^2 = f^2(\theta)}$

FPs $f(\theta)=0$ $\quad (f(\theta))^2 = 0 \iff f(\theta) = 0$

$\therefore$ FP sets are the same.

Flow between consecutive fixed points can be positive or negative for $\dot{\theta} = f(\theta)$ dep. on function f but flow of $\dot{\theta} = (f(\theta))^2$ is always $\dot{\theta} > 0$, i.e anticlockwise

$\therefore$ Flows reversed to anticlockwise if clockwise is the only change to the phase portrait of $\dot{\theta} = (f(\theta))^2$ by comparison with $\dot{\theta} = f(\theta)$

Question 3 [40 marks]. Two dimensional systems

(a) Consider the system,

$$\dot{x} = x(1 - 2y) \quad \dot{y} = -y(1 - x). \tag{3}$$

(i) Find the fixed points of system (3) and classify them, sketch the null-clines and the vector field for the positive quadrant. **[6]**

(ii) For which fixed points of system (3) does the Hartman-Grobman theorem assist in identifying their types of stability? **[4]**

(iii) Find a first integral for system (3) of the form $f(x) + g(y) = C$, a constant. By examining the form of each of the functions $f(x), g(y)$, or otherwise, determine the nature of all the fixed points and sketch the phase portrait in the first quadrant. **[6]**

$$\dot{x} = x(1-2y) \qquad \dot{y} = -y(1-x)$$

FPs given by $\dot{x} = \dot{y} = 0$: $(x,y) = (0,0)$ $(x,y) = (1, \tfrac{1}{2})$

Jacobian at (x,y) is $J = \begin{bmatrix} \frac{\partial}{\partial x}(\dot{x}) & \frac{\partial}{\partial y}(\dot{x}) \\ \frac{\partial}{\partial x}(\dot{y}) & \frac{\partial}{\partial y}(\dot{y}) \end{bmatrix} = \begin{bmatrix} 1-2y & -2x \\ y & x-1 \end{bmatrix}$

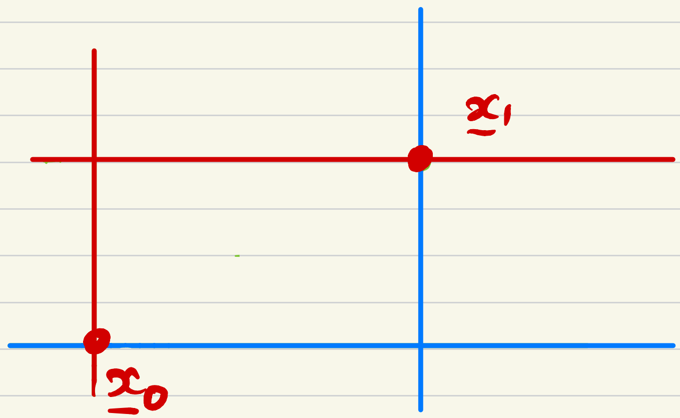
$\underline{x}_0 = (0,0)$ $J[\underline{x}_0] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\lambda = \pm 1$ – saddle $\lambda_1 = +1$, $\underline{v}_1 = (1,0)$

$\lambda_1 = -1$ $v_2 = (0,1)$

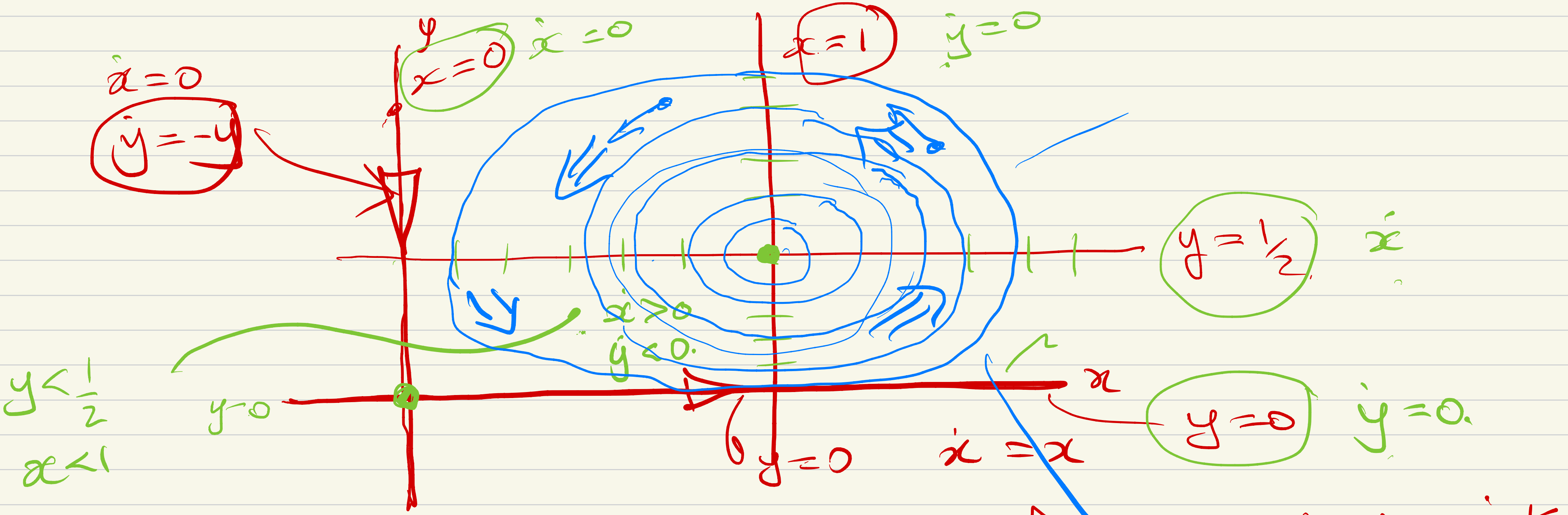
$\underline{x}_1 = (1, \tfrac{1}{2})$ $J[\underline{x}_1] = \begin{bmatrix} 0 & -2 \\ \frac{1}{2} & 0 \end{bmatrix}$ $\lambda = \pm i$ – linear centre no real eigenvalues

<u>Null clines</u> $\dot{x} = 0$ $x = 0$ $y = \tfrac{1}{2}$

$\dot{y} = 0$ $y = 0$ $x = 1$



Sketches and adding information on the system in the (x,y) plane live! → see cleaner version on next pages!



$\dot x = x(1-2y)$ $(0, y)$ $\dot x = 0$ move ↕ y-axis invariant

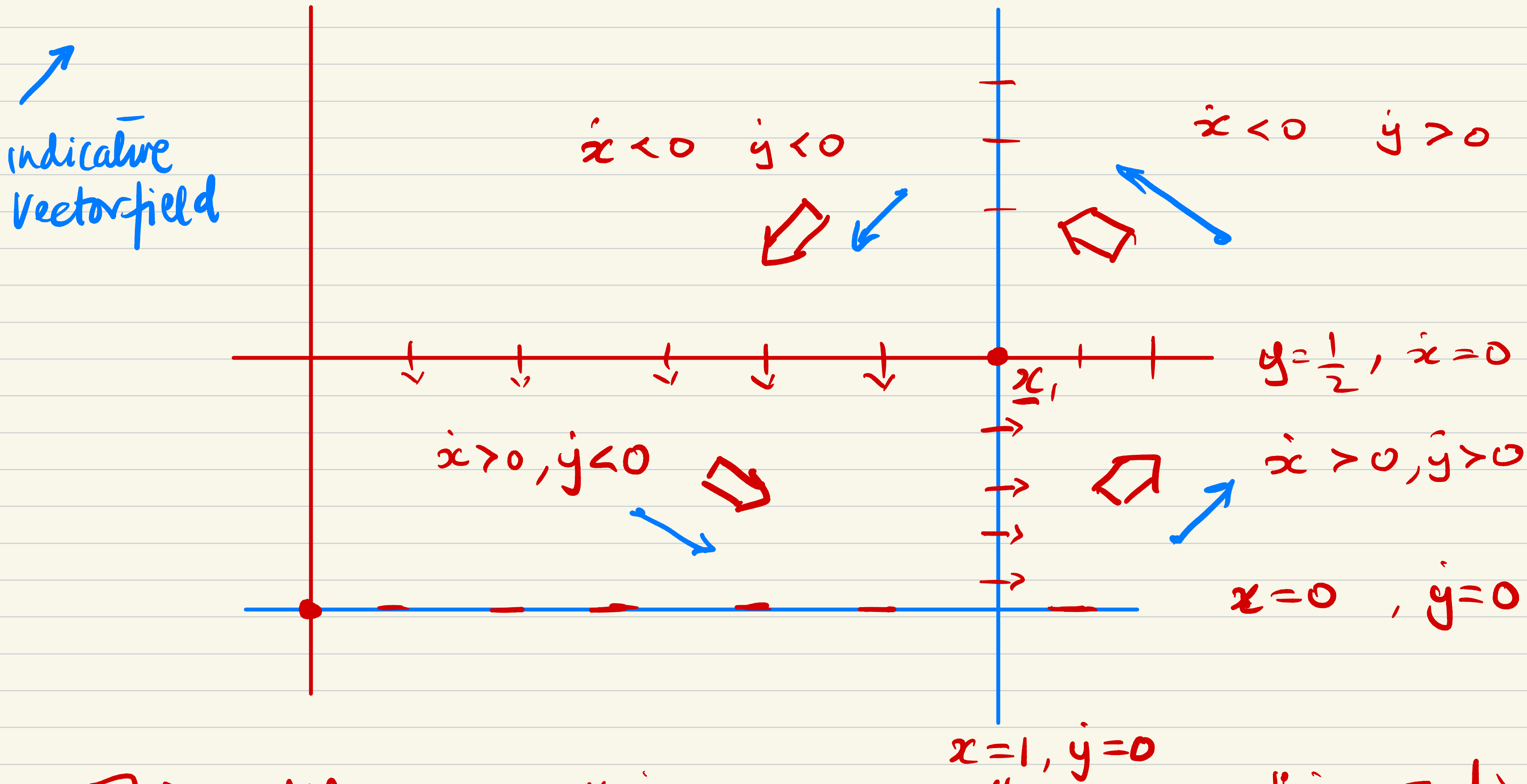
$\dot y = -y(1-x)$ $(x, 0)$ $\dot y = 0$ move ↕

linear centre

And also a non-linear centre as a result of $f(x) + g(\ ) = c$.

$g(y)$ ...... max

where $f(x)$ has unique max



This reflects the "circular motion" of the linearisation at $\underline{x}_1^*$, question what is the non-linear behavior?

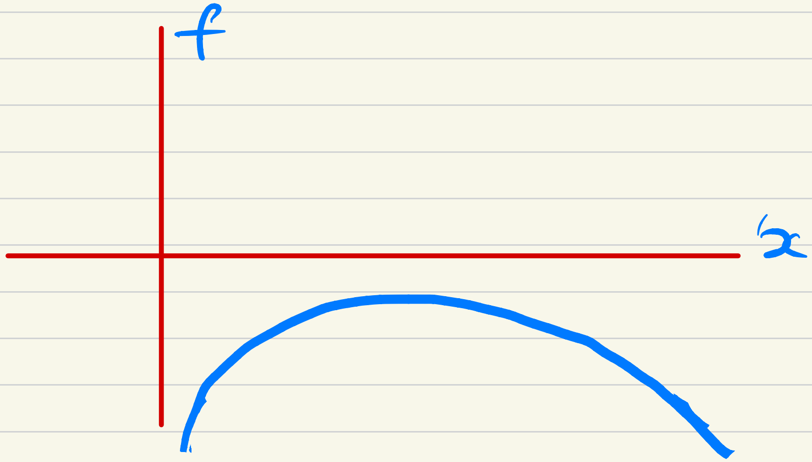
Asked to find 1st integrals of the system

$$\frac{dx}{x(1-2y)} = \frac{dy}{-y(1-x)}$$

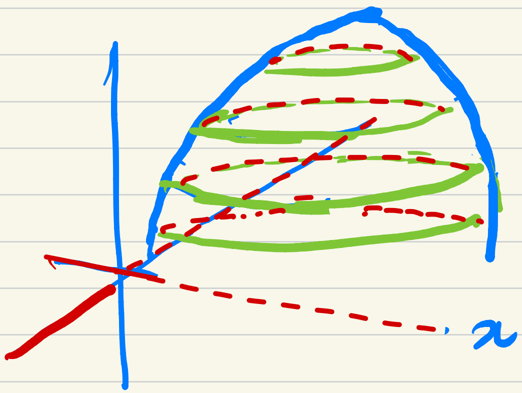
$$\Rightarrow \int \frac{1-x}{x}\,dx + \int \frac{1-2y}{y} = 0$$

$$\Rightarrow \underbrace{\ln x - x}_{f(x)} + \ln y - 2y = \text{const}$$

$f(x) = \ln x - x$ for $x > 0$



So we have a surface $z = f(x) + g(y)$ with a max at $x = 1, y = \frac{1}{2}$



$\therefore$ $z = \text{const}$ gives concentric closed curves around the maximal point.

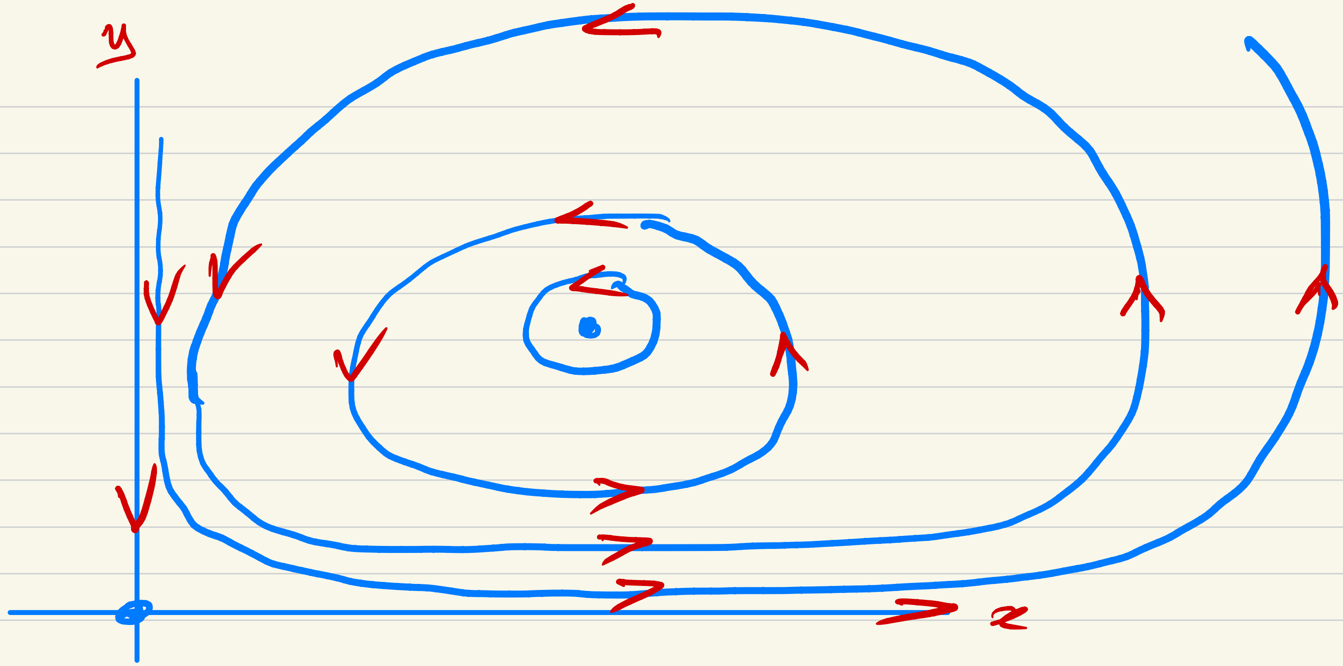
$f'(x) = \frac{1}{x} - 1$ $\therefore$

$f'(x) = 0$ at $x = 1$

$f''(x) = -\frac{1}{x^2}$ $\therefore$ max and < 0

Similar for $g(y)$ $\therefore$ max at $y = \frac{1}{2}$

$\therefore$ The non-linear system has not only a linearized centre but a non-linear centre.



Concluding qualitative phase portrait showing key features for the given system.

Note $f(x) = x - \ln(x)$ and $g(x) = 2y - \ln(y)$ would be a function that might be considered as the first integral, then you get a global MINIMUM at $x=1, y=\frac{1}{2}$ but conclusion on closed periodic orbits remain the same.