DYNAMICAL SYSTEMS

MTH744 U/P

SEMESTER A

2023 – 2024

David Arrowsmith

WEEKS 1 - 4

Introductory Remarks

QMplus set up

- Latex lecture notes of module
- Week by week indication of content
- "Quizzes" and "Examples"
    - do not count towards final mark.
- Past exam papers and solutions
- "Examples" solutions
- Quizzes can be repeated ad-infinitum!

Book - Strogatz

Office Hour for MTH744U/P

12.30 to 2.00 pm

Email: d.k.arrowsmith@qmul.ac.uk

Remote meetings on Teams

Lectures 2hr/wk for 11 weeks (exc week 7)

Mid-term Test in Week 7

Final Examination (Jan '24) 80%

Mid Term (Nov '23) 20%

Dynamical Systems on $\mathbb{R}$

$x = x(t)$, t independent variable
x depends on

ODEs in one dependent variable
equation involving derivatives of x wrt t and x and t

E.g.

$\frac{dx}{dt} + x = 0$ order 1

$x\frac{dx}{dt} + x + \sin(xt) = 0$ order 1

$\frac{d^2x}{dt^2} + x\frac{dx}{dt} + t^2x = 0$ order 2

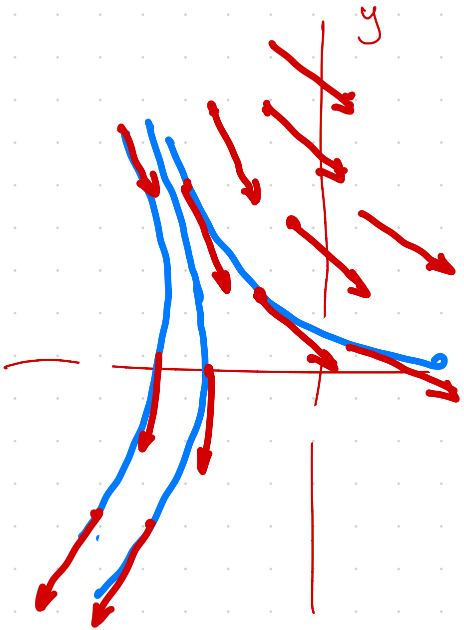
$\frac{dx}{dt} + x = 0 \Rightarrow \frac{dx}{dt} = -x$, general form for W1.2

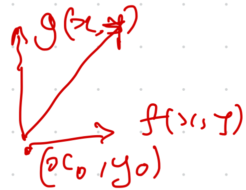
1st order eqn in 1 DV is $\frac{dx}{dt} = f(x)$

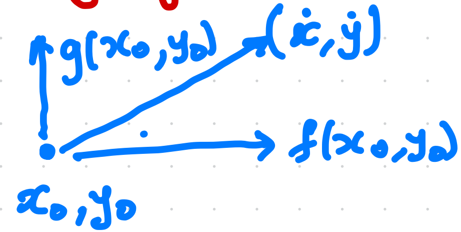
$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + x^3 = 0$$

W1.3

$\dot{x} = f(x,y)$, $\dot{y} = g(x,y)$







integral curves
solutions
orbits are The Same
thing!

Wolfram alpha APP try "vectorfieldplot {y, -x}"

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x$$

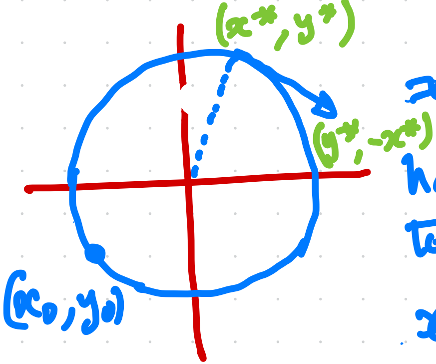
$$\Rightarrow dt = \frac{dx}{y} = \frac{dy}{-x}$$

$$\Rightarrow x\,dx + y\,dy = 0$$

$$\Rightarrow x^2 + y^2 = \text{const.} = x_0^2 + y_0^2$$ if orbit passes through (x_0, y_0)

Integral curves of $\dot{x} = f(x,y)$, $\dot{y} = g(x,y)$ are ones which are tangent to the v.f. at every point (x,y)

W1.4

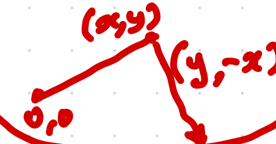


$\dot{x} = y, \ \dot{y} = -x$

has solutions constrained to concentric circles

$x(t)^2 + y(t)^2 = x_0^2 + y_0^2$

So at each pt (x, y), we have the vector $(y, -x)$



$x^*, y^* > 0 \Rightarrow \frac{dx}{dt} = y^* > 0 \quad \frac{dy}{dt} = -x^* < 0$ SE

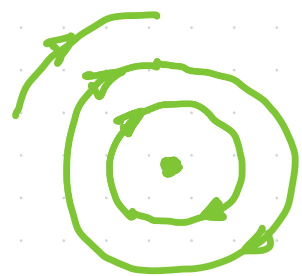
$+$ $-$

1st quad SE

2nd quad : $(+,+)$ ↗

3rd quad : $(-,+)$ ↖

4th quad : $(-,-)$ ↙



periodic motion
periodic orbit
periodic soln curve

1st order systems on $\mathbb{R}$

$\dot{x} = f(x)$, $f : \mathbb{R} \to \mathbb{R}$

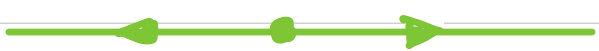
Qualitative dynamics - what "form" does an orbit take and what are its asymptotics?

Quantitative dynamics is concerned with finding exact orbit solutions

Qualitative dynamics is completely solvable on $\mathbb{R}$ whereas Quantitative dynamics isn't!

e.g. $\dot{x} = x + \sinh(x) = f(x)$ is given by a globally unstable fixed point

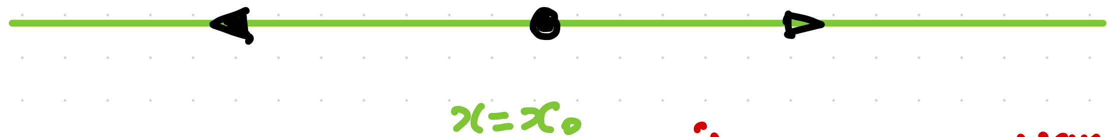
$$x + \sinh(x) \begin{cases} > 0 & \text{for } x > 0 \\ = 0 & x = 0 \\ < 0 & x < 0 \end{cases}$$



whereas $\int \frac{dx}{x + \sinh(x)}$ has no result in standard mathematical functions!

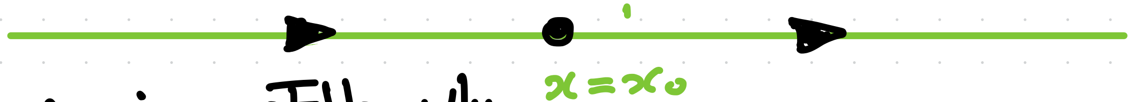
Consider The phase portrait

e.g. $\dot{x} = x$



- unstable, orbits on both sides move away from $x = x_0$ as t - increases.

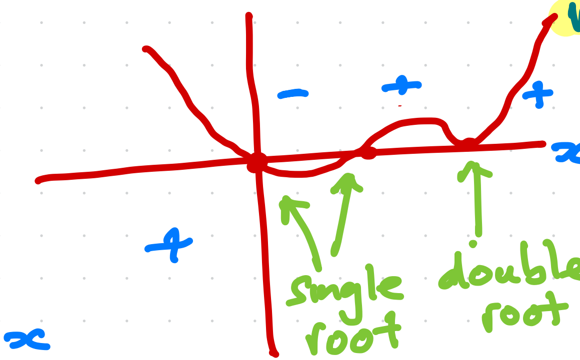
e.g $\dot{x} = x^2$?



- Again unstable - why

- stable fixed point - all orbits move closer to $x = x_0$ as t - increases

$\dot{x} = f(x) = x(x-1)(x-2)^2$ gr(f)



not LU

linearly unstable = LU

linearly stable LS

Note fixed pts are solns of ODE!

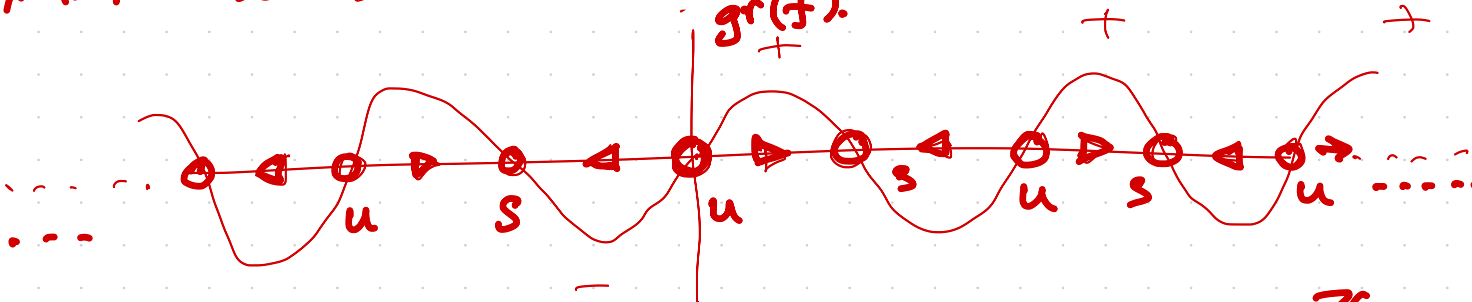
i.e. $x(t) \equiv 0$ is a fixed pt solution

and $\frac{dx}{dt} = 0 = f(0)$. Same for $x \equiv 1$, $x \equiv 2$

W1.8

Note $\dot{x} = \frac{1}{x-1}$ is not defined for $x=1$, therefore we have two dynamical systems for $x<1$ & $x>1$!

Ex 1.7 $\dot{x} = \sin(x)$ $(= f(x))$ F.P.s when $\sin(x)=0$, $x = n\pi$, $n \in \mathbb{Z}$



→ ordered sequence of fixed pts at $x = n\pi$, $n \in \mathbb{Z}$ alternating between stable and unstable points

Note $f'(x) = \cos x \Rightarrow \cos(n\pi) = (-1)^n$ – alternate + and −

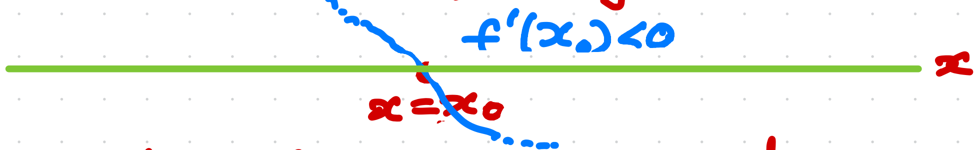
<u>Quantitatively</u> $\dot{x} = \sin x \Rightarrow \int \operatorname{cosec} x \, dx = dt$

$\Rightarrow -\log(\operatorname{cosec} x + \cot x) = t + C$

but $x = F(t)$, F ??

Linear Stability

$\dot{x} = f(x)$ – properties of f can determine stability



The graph of $f(x)$ has a zero at $x = x_0$, i.e. $f(x_0)=0$.

Suppose $f'(x_0)$ is negative then in a small nbd of x_0

if $x < x_0$, $f(x) > 0$ & $x > x_0$, $f(x) < 0$

$\therefore$ STABILITY

Similarly if $f'(x_0) > 0$, we get INSTABILITY

LINEAR INSTABILITY	$f'(x_0) > 0$
LINEAR STABILITY	$f'(x_0) < 0$

Week 2 - Lectures 3 - 4

- Note on dynamical system in n-dimensions
- Linear stability at a fixed point Linearisation. Example 1.7
- Def^n of stability / asymptotically stable / unstable

(leftovers from Chapter 1)

Bifurcations of systems on the line $\mathbb{R}$ (Chapter 2).

W2.2

Dynamical systems in n-dimensions

$n=1$, $\dot{x} = f(x)$; $n=2$, $\dot{x} = f(x,y)$, $\dot{y} = g(x,y)$; *

$n=3$, $\dot{x} = f(x,y,z)$, $\dot{y} = g(x,y,z)$, $\dot{z} = h(x,y,z)$.

general n

$\underline{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ → n-dependent variable

$\underline{f} : \mathbb{R}^n \to \mathbb{R}^n$ velocity vector

$\underline{f}(\underline{x}) = (f_1(\underline{x}), f_2(\underline{x}), \ldots, f_n(\underline{x}))$, each $\in \mathbb{R}$

So the position $\underline{x} = (x_1, x_2, \ldots, x_n)$ has a vector $\underline{f}(\underline{x})$ attached to it cf * $\underline{x} = (x_1, x_2, x_3) = (x, y, z)$ and $(f_1, f_2, f_3) = (f, g, h)$ in notation above.

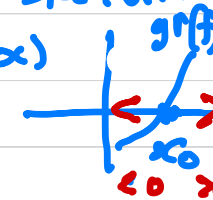
$$\boxed{\text{We only consider } n=1 \text{ \& } n=2}$$

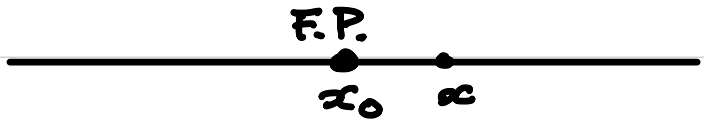
$$\boxed{\dot{\underline{x}} = \underline{f}(\underline{x})}$$

Local behaviour at a fixed point of $\dot{x}=f(x)$ at $x=x_0$ (i.e. $f(x_0)=0$).

Linear Stability

$\dot{x}=f(x)$





Let $x = x_0+\eta$ (η is a local coordinate)

Substitute in $\dot{x}=f(x)$ gives ($x(t)=x_0+\eta(t)$)

$$\dot{x} = \dot{\eta} = f(x_0+\eta)$$

$$= \underset{\approx ?}{} \underbrace{f(x_0)}_{=0} + f'(x_0)\underbrace{(x-x_0)}_{\text{"}\eta} + \cancel{\frac{f''(x_0)(x-x_0)^2}{2!}}$$

i.e.

$$\dot{\eta} = f'(x_0)\,\eta \qquad \text{(linear system)}$$

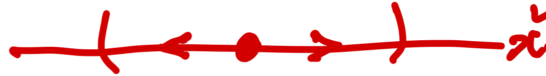
Linear system approximation

$f'(x_0)<0$ stable fixed point

$f'(x_0)>0$ unstable fixed point

$f'(x_0)=0$ UNSURE!

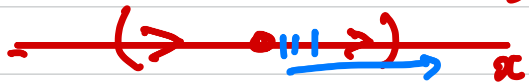
Stability



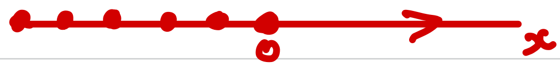
✓ U FP

Neighbourhood N

unstable



✓ U FP



U ✓ FP

$\dot{x} = 0 \quad x \le 0$

$\dot{x} > 0 \quad x > 0$

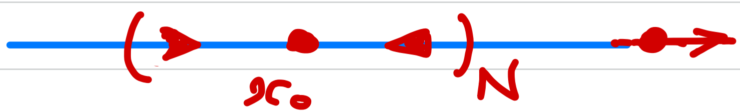
stability

✓ S FP

$x(t)$, $x(0) = x^*$

$x(t) \in N \ \forall t > 0$

stability for pts $x^* \in I$



✓ S FP

$\dot{x} \equiv 0$ ✓ S FP.

W2.5

$$\left[\ddot\theta = -\sin\theta \qquad \left.\begin{aligned}\dot\theta &= v\\ \dot v &= -\sin\theta\end{aligned}\right\}\right]$$

Asymptotic stability $\rightarrow$ stability plus $x(t)\longrightarrow x_0$ as $t\nearrow$

$$\dot x = 1+x^2 \qquad \int\frac{dx}{1+x^2} = \int dt$$

$$\arctan(x) = t + c$$

$$\Rightarrow x = \tan(t+c),$$

Exp Let $c=0$, note

$x\rightarrow\infty$ as $t\rightarrow\pi/2$ i.e. x escapes to ∞ as $t\longrightarrow\pi/2$

W2.6

Bifurcation Theory

$\dot{x} = f(x)\ ,\ x \in \mathbb{R}$

$\dot{x} = r + x^2\ ,\ x \in \mathbb{R}\ ,$ r is a parameter

$\dot{x} = ax - bx^2\ ,\ a\ ,\ b$

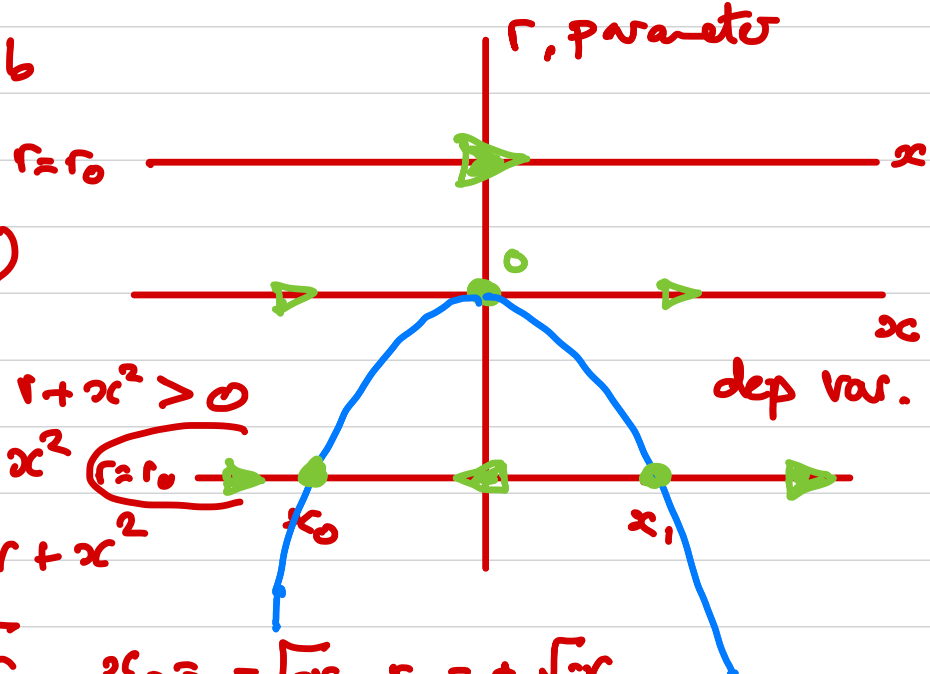
$\dot{x} = r + x^2 = f(r, x)$

$r = r_0$

$r > 0\ :\ \dot{x} = r + x^2 > 0$

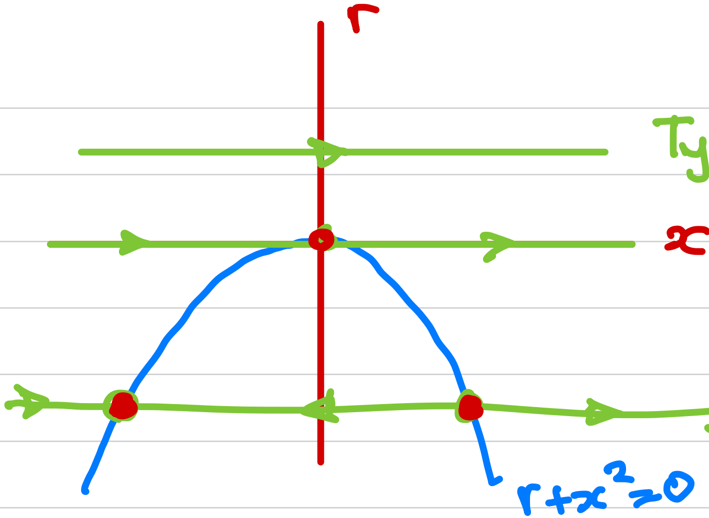
$r = 0\ :\ \dot{x} = x^2$

$r < 0\ :\ \dot{x} = r + x^2$



FP

$r + x^2 = 0\ ,\ x = \pm\sqrt{-r}\ ,\ x_0 = -\sqrt{-r}\ ,\ x_1 = +\sqrt{-r}$



Type 1

Type 2

Type 3

Dynamical structure or qualitative type changes at $r=0$

Bifurcation at $r=0$

Saddle-node bifurcation — archetypal example

Bifurcation diagram in xr - plane

$\dot{x} = r + x^2 \quad \dot{r} = 0.$

"phase portrait" in 2D

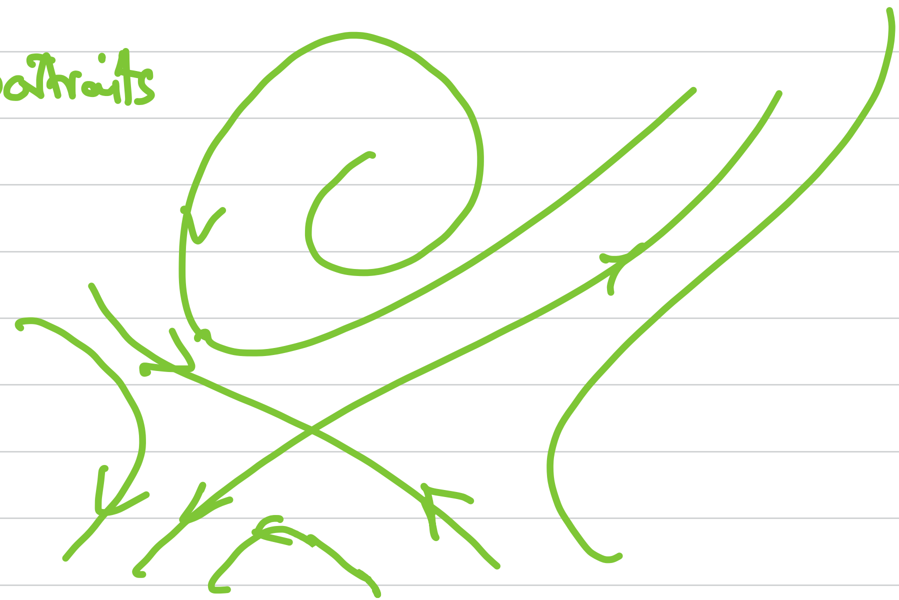
Very special one!!

w2,8

B.f. D'iag always looks like "horizontal line"

dynamics in 2D

Phase portraits typically looks different!



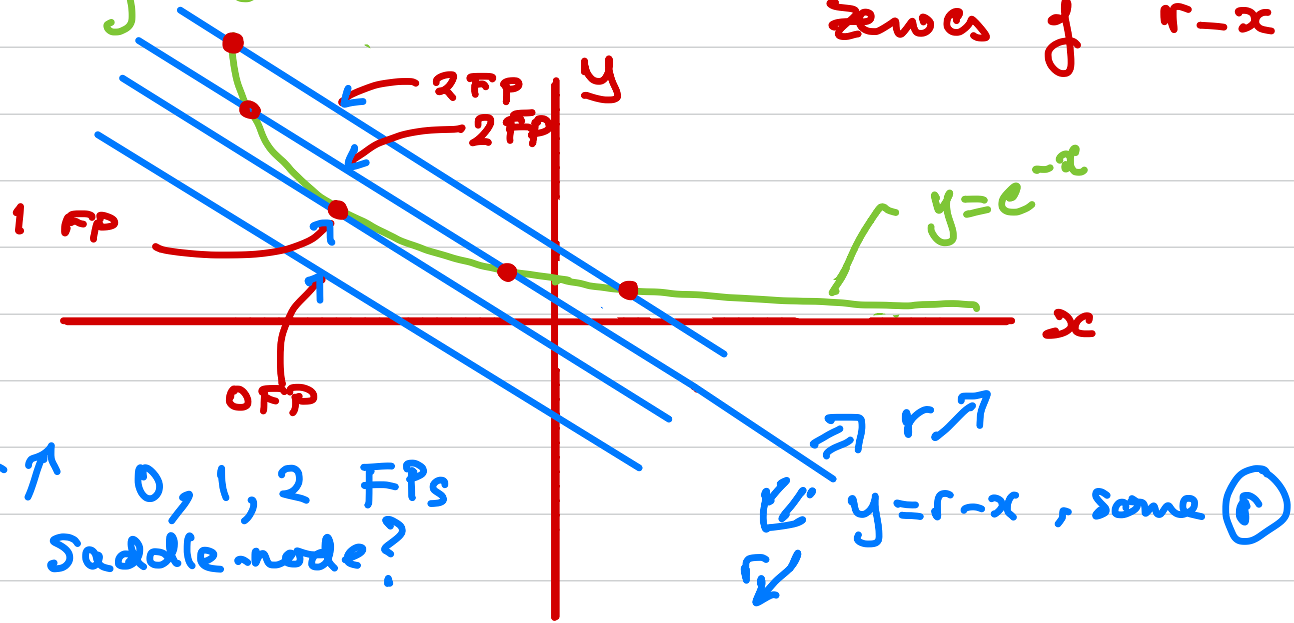
Ex 2.3

$$\dot{x} = r - x - e^{-x} \qquad \text{FPs}$$

$$r - x - e^{-x} = 0 \quad \Rightarrow ??$$

$$\left.\begin{array}{l} y = r - x \\ y = e^{-x} \end{array}\right\} \text{plot these:}$$

intersection of these graphs will gives zeros of $r - x - e^{-x} = 0$



r ↑ 0, 1, 2 FPs

Saddle-node?

$$r - x - e^{-x} = 0$$

$$f(r,x) = r - x - e^{-x}$$

$$\left.\begin{aligned} f(r,x) &= 0 \\ \frac{\partial f}{\partial x}(r,x) &= 0 \end{aligned}\right\} \text{ ensures tangency}$$

$$\Rightarrow \left.\begin{aligned} r - x - e^{-x} &= 0 \\ -1 + e^{-x} &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} &r - 0 - 1 = 0 \\ &e^{-x} = 1, \; x = 0 \end{aligned}$$

$x = 0, r = 1$ is a potential bifurcation

$$f(r,x) = r - x - e^{-x}$$

$$= r - \cancel{x} - \left(1 - \cancel{x} + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\right)$$

$$= (r-1) - \frac{x^2}{2!} + \frac{x^3}{3!} \dots\dots$$

Local coordinates - reduction to normal form

$\mu = r - 1, \quad y = x$

$\dot{x} = \dot{y} = \mu - \frac{x^2}{2} + \dots \Leftarrow \mu - \frac{y^2}{2}$

$= \mu - \frac{y^2}{2} + \dots$

$1 \quad \frac{1}{2}$

Let $z = \alpha y$ $\quad \alpha$?, so substitute!

$\dot{z} = \alpha \dot{y} = \alpha\left(\mu - \frac{z^2}{2\alpha^2}\right)$

Changes of coords to reduce to "NORMAL FORM"

$\dot{z} = \alpha\mu - \frac{z^2}{2\alpha}$ $\quad$ Choose $\alpha = -\frac{1}{2}$.

$\dot{z} = -\frac{\mu}{2} + z^2$ $\quad$ $\nu = -\frac{\mu}{2}$

$\boxed{\dot{z} = \nu + z^2 + \dots} \Rightarrow z \approx \pm\sqrt{-\nu}$

Saddle-node normal form.

Saddle node normal form

$$\dot{x} = f(x, r)$$

$$\downarrow$$

Taylor

$$\left.\begin{aligned} f(x, r) &= 0 \\ \frac{\partial f}{\partial x}(x, r) &= 0 \end{aligned}\right\} \text{double root}$$

$$(x, r) = (x_0, r_0)$$

$x = x_0$ $\quad f(x) = 0$

$f'(x) = 0$

$$f(x) = (x - x_0)^2 g(x)$$

$$f(x,r) = \underbrace{\cancel{f(x_0, r_0)}}_{=0} + \underbrace{\cancel{\frac{\partial f}{\partial x}(x_0, r_0)}}_{=0}(x - x_0) + \underbrace{\frac{\partial f}{\partial r}(x_0, r_0)}_{=A}(r - r_0)$$

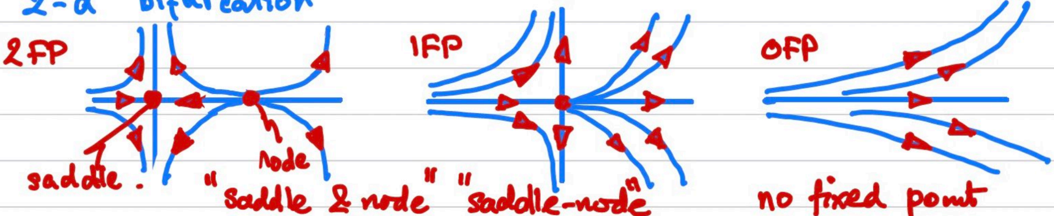
$$+ \underbrace{\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x_0, r_0)}_{=B}(x - x_0)^2 + \underbrace{\frac{\partial^2 f}{\partial x \partial r}(x_0, r_0)}_{=C}(x - x_0)(r - r_0)$$

$$+ \frac{1}{2}\frac{\partial^2 f}{\partial r^2}(r - r_0)^2 + \ldots$$

In the two examples we have $A = \frac{\partial f}{\partial r}$ and $B = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}$ at the bifurcation point (x_0, r_0) are both non-zero. This means that the bifurcation pt is of saddle-node type.

the fixed point at which the bifurcation occurs is said to be a "saddle-node"

The phrase saddle-node arises from the corresponding 2-d bifurcation



W3.1

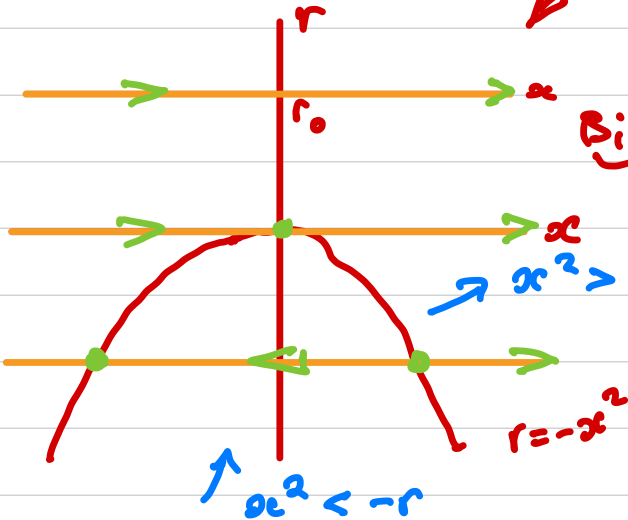
Saddle-node

$\dot{x} = r + x^2$

$(\dot{x} = r - x^2)$

$\dot{y} = r - y^2$

$x = -y$??



Bifurcation diagram

FP set

FP stability ? $\dot{x} = r + x^2 = f(x,r)$

FPs $x = \pm\sqrt{-r}$ Linear stability at FPs.

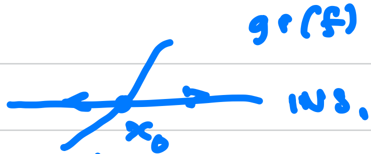
$$\left.\frac{\partial f}{\partial x}(x,r)\right|_{x=+\sqrt{-r}}$$

$$= 2x\Big|_{x=+\sqrt{-r}}$$

$$= 2\sqrt{-r} > 0$$

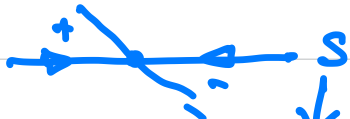
"linear" unstable at $x = +\sqrt{-r}$

"linear" stable at $x = -\sqrt{-r}$

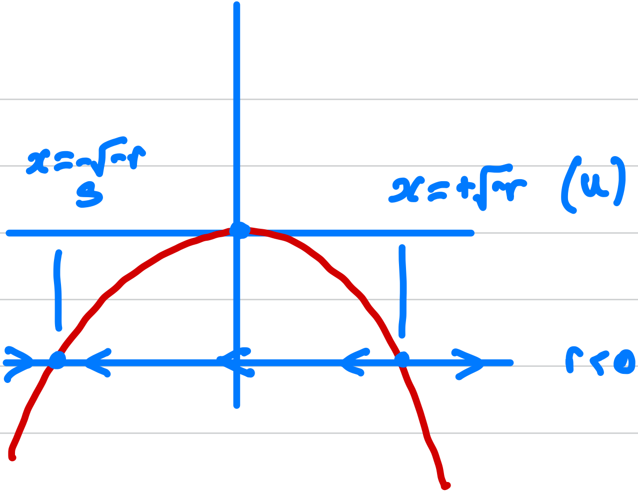


$\frac{df}{dx} > 0$

$\frac{df}{dx} < 0$



Asymptotically Stable



$r=0$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = 0$$

More investigation

$\dot{x} = x^4 = f_1(x)$

$\dot{x} = x^3 = f_2(x)$

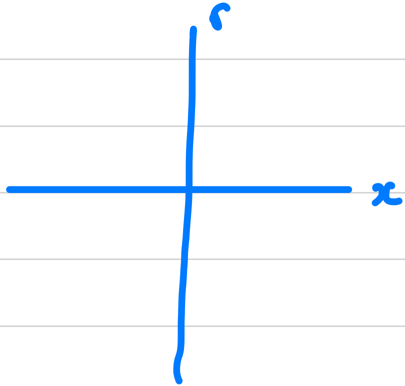
$$\frac{\partial f_1}{\partial x} = \left.\frac{\partial f_2}{\partial x}\right|_0 = 0$$

N3.4

$\dot{x} = r - x - e^{-x}$ $\quad r=1, x=0$ FP.

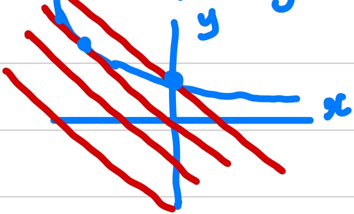
FPs at $r - x - e^{-x} = 0$

$r + x^2 = 0$



$r - x = e^{-x}$

$y = r - x$, $y = e^{-x}$



$f(x,r)$

$f_r(x)$

Transcritical Bifurcation

$\dot{x} = xr - x^2$ — canonical model $C=1, B=-1$

cf. $\dot{x} = r + x^2$

Taylor expansion
A, B, C

SNB A, B ✓
TB B, C ✓

$A = \left.\frac{\partial f}{\partial r}\right|_{(x^*, r^*)}$ $B = \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x^*, r^*)}$

$C = \left.\frac{\partial^2 f}{\partial x \partial r}\right|_{(x^*, r^*)}$

$f(x, r) = Ar + Bx^2 + Cxr +$ H.O.T.

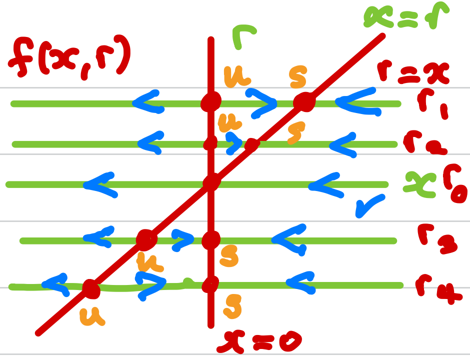
Investigate $\dot{x} = xr - x^2 = f(x,r)$

FP set $xr - x^2 = 0$

$x(r-x) = 0$

FP lines $x \equiv 0$

$r = x$



$\dot{x} = \frac{dx}{dt}$

FP sequence $2 \to 1 \to 2$ as $r \nearrow$

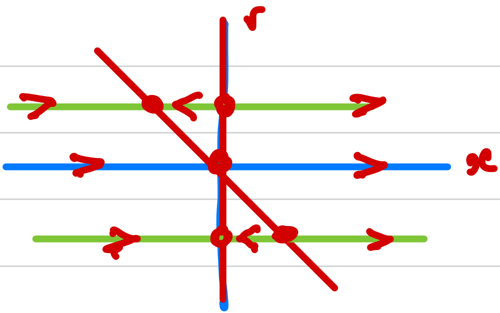
Lin Stability calculation

$$\left.\frac{\partial f}{\partial x}(x,r)\right|_{(0,r)} = \left. r - 2x \right|_{x=0} = r \begin{cases} r>0 & \text{unstable} \\ r<0 & \text{stable} \end{cases}$$

w3.7

$$\left.\frac{\partial f}{\partial x}\right|_{(r,r)} = r - 2x \Big|_{r=x} = r - 2r = -r \quad \begin{cases} r>0 \text{ stab.} \\ r<0 \text{ unstable} \end{cases}$$

Note $\dot{x} = rx + x^2 = x(r+x)$



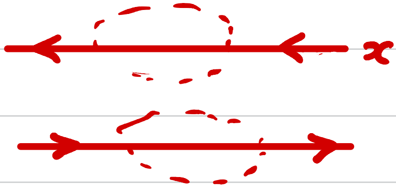
$|x| \gg 1 \quad \dot{x} \gg 0$

Mathematically, up to a change of coordinates, they are exactly the same system

$\dot{x} = rx - x^2$, let $y = -x$

$\dot{y} = -\dot{x} = -rx + x^2 = ry + y^2$

so $\dot{y} = ry + y^2 \iff \dot{x} = rx - x^2$



transformation $x \to y$

changes orientation

→ careful interpretation

Ex 2.6 p15 Lecture notes. w3.9

$\dot{x} = r\ln(x) + x - 1 = f_r(x) = f(x,r)$

Note $x = 1 \Rightarrow \dot{x} = 0 \quad \forall r$ $\quad \ln(1) = 0$

$x - 1 = 0$

FP set By observation $x \equiv 1$ F.P set in (x,r) plane

$x = y + 1$, y = local coord at the FP $x = 1$ $\forall r$

$\dot{y} = \dot{x} = r\ln(x) + x - 1$

$$\boxed{\dot{y} = r\ln(1+y) + y}$$

$$= r\left(y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \dots\right)$$

i.e. $r\left(y - \frac{y^2}{2} + \cdots\right) + y = 0$

$$\dot{y} = (r+1)y - r\frac{y^2}{2} + \cdots$$

$\dot{x} = rx - x^2$

$$\dot{y} = \mu y + (1-\mu)\frac{y^2}{2}$$

$r+1 = \mu$

$-r = 1-\mu$

$$= \boxed{\mu y + \frac{y^2}{2}} - \mu\frac{y^2}{2} + \text{H.O.T.}$$

let $z = \alpha y$ $\quad \alpha$? $\quad \alpha = \frac{1}{\sqrt{2}}$???

$$\dot{z} = \alpha\dot{y} = \cancel{\alpha}\mu\frac{z}{\cancel{\alpha}} + \frac{1}{2}\left(\frac{z}{\alpha}\right)^2$$

Note $C \neq 0$, $B \neq 0$,

See Latex notes p12, last line

Transcritical form

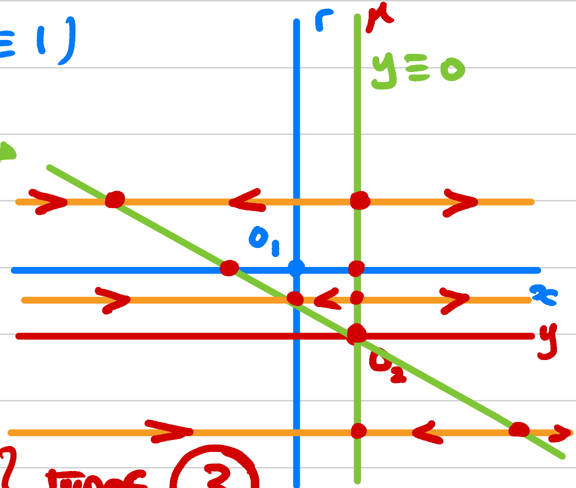
$\bullet$ = FP

$$\dot{y} = \mu y + \frac{y^2}{2}$$

$C = 1 \neq 0$, $B = 1/2 \neq 0$

FP curves $y \equiv 0$ $(x \equiv 1)$

$\mu = -y/2$



$y = 0 \Leftrightarrow x = 1$

$\mu = 0$, $r = -1$

$(y, \mu) = (0, 0)$

corresponds to

$(x, r) = (1, -1)$

# of distinct {topological / qualitative} types 3 — 2 - 1 — 2

Taylor expansion

W4.1

$$\dot{x} = f(x,r) = f(x_0,r_0) + \frac{(x-x_0)}{1!} \overset{=0}{f_x(x_0,r_0)} + \ldots$$

$$\overset{A}{+ \frac{(r-r_0)}{1!} f_r(x_0,r_0)} + \frac{(x-x_0)^2}{2!} \overset{B}{f_{xx}(x_0,r_0)}$$

$$+ \frac{(x-x_0)(r-r_0)}{1!} \overset{C}{f_{xr}(x_0,r_0)} + \frac{(r-r_0)^2}{2!} \overset{D}{f_{rr}(x_0,r_0)}$$

$$+ \frac{(x-x_0)^3}{3!} \overset{E}{f_{xxx}(x_0,r_0)} + \ldots$$

$A, B \neq 0$, C?
Saddle node

&

$A = 0$, $B, C \neq 0$
Transcritical

Third bifurcation to consider
PITCHFORK BIFURCATION

W4.2

Pitchfork bifurcation

Consider $f(x,r) = -f(-x,r)$

oddness in x, not in r.

$x,\ x^3,\ x^5, \dots$

$f(x) = x^2$

$f(-x) = x^2$

$f(x) + -f(-x)$

$C=1 \quad E=1$

$\dot{y} = f(y,\mu) = \mu y + y^3$

more generally

$= C\mu y + Ey^3 + D\mu^2 ? + \dots$

(y,μ) local coordinates at (x_0, r_0)

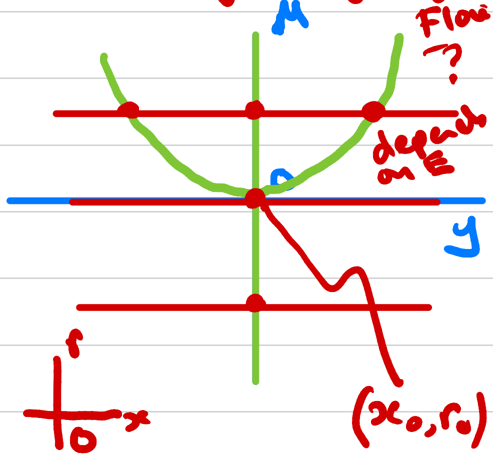
FP set $\quad C\mu y + Ey^3 + D\mu^2 + F\mu y^3 = 0$

$C\mu y + Ey^3 = 0$ neglecting H.O.T.

$y(C\mu + Ey^2) = 0$

$y = 0 \qquad y = \pm\sqrt{\frac{-C\mu}{E}}$

$\dot{y} = C\mu y + Ey^3$



$\mu = -\frac{E}{C}y^2$

$-\frac{E}{C} > 0$ ✓ $\cup$ super-critical

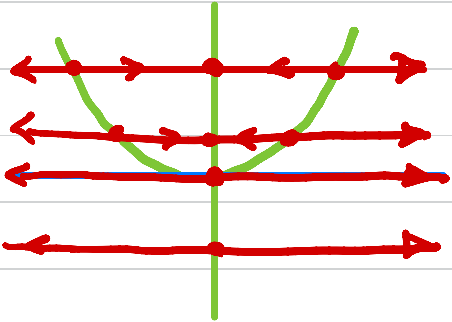
$-\frac{E}{C} < 0$ $\cap$ sub-critical

supercritical pitchfork $1 \to 1 \to 3$ on $\mu \nearrow$

W4.4

Subcritical $3 \to 1 \to 1$ as $\mu \nearrow$.

$\dot{y} = C\mu y + E y^3$, $E > 0$, $-\frac{E}{C} > 0$ ✗



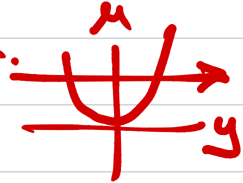
Supercritical
V1: $E>0$; V2: $E<0$

Subcritical
V3: $E>0$, V4: $E<0$

Extra terms solve for FP leading linear term μ + h.o.t (e.g. μ^2 etc.)

$$\dot{y} = -3\mu y + 1.y^3$$

F.P. set $y = 0$, $3\mu = y^2$.



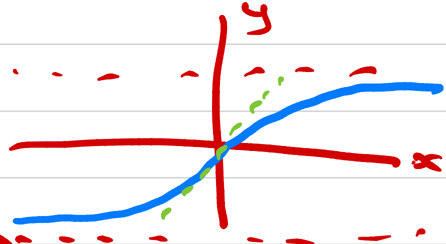
Supercritical

Ex. 2.7.

$$\dot{x} = -x + \beta \tanh(x)$$

$y = x$, $y = \beta \tanh(x)$

where graphs intersect give fixed pt.

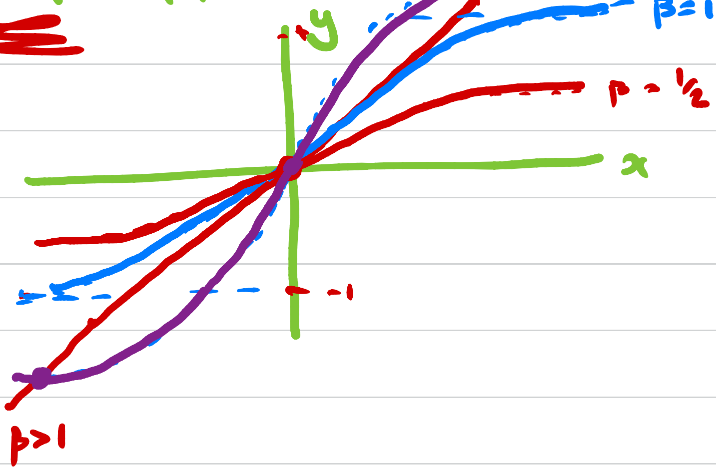


Rev of $\tanh(x)$!

$y = \tanh(x)$

$y' = \operatorname{sech}^2 x = \dfrac{1}{\cosh^2 x}$

NOT BIFURCATION DIAGRAM



but an attempt to understand fixed pt pattern as $\beta \to$ increases thru $\beta = 1$

W4.7

$$\dot{x} = -x + \beta\tanh(x) \qquad = f(x,\beta)$$

$$= -x + \beta\left(x - \frac{1}{3}x^3 + \frac{2}{15}x^5 \ldots\right)$$

$$= (\beta-1)x - \beta\frac{x^3}{3} + \beta\frac{2}{15}x^5 + \ldots$$

$$= (\beta-1)x - \frac{1}{3}\beta x^3$$

For Bif Pt

$f(x,\beta)=0$

$\frac{\partial f}{\partial x}(x,\beta)=0$

$\Updownarrow$

$\beta - 1 = 0$

$x = 0.$

Let $\mu = \beta - 1$

$$\dot{x} = \mu x - \frac{1}{3}(\mu+1)x^3 + \ldots$$

$$= \mu x - \frac{1}{3}x^3 - \frac{1}{3}\mu x^3$$

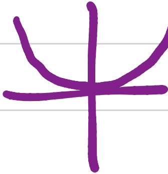
$$= x\left(\mu - \frac{1}{3}x^2 - \frac{1}{3}\mu x^2\right)$$

FP? $\longrightarrow$

$$\mu = \frac{1}{3}x^2 + \mu\frac{1}{3}x^2$$

$$\mu \approx \frac{1}{3}x^2 +$$

Why?

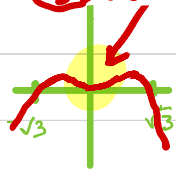


close to parabola

Note

$$\mu = \frac{1}{3}x^2$$

locally $\frac{1}{3}x^2$



$$\mu\left(1-\frac{1}{3}x^2\right) = \frac{1}{3}x^2$$

$$\mu = \frac{1}{3}x^2\left(1-\frac{1}{3}x^2\right)^{-1}$$

$$= \frac{1}{3}x^2\left(1+\frac{1}{3}x^2\right) = \frac{1}{3}x^2+\frac{1}{9}x^4$$

↑ graph

$$\approx \frac{1}{3}x^2$$

FP

Ex. 2.8

$\dot{x} = x^5 - x^3 - rx$

FP set : $x^5 - x^3 - rx = 0$

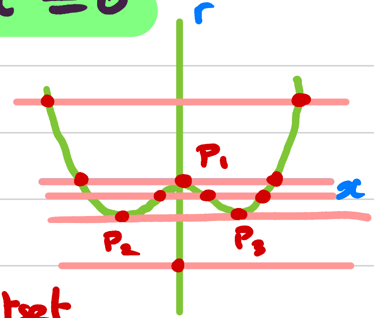
TRUE, BUT! $x \neq 0$ — $r = \dfrac{x^5 - x^3}{x}$

BETTER! $x(x^4 - x^2 - r) = 0$

$x = 0$

$r = x^2 - x^4$

} Fixed point set → Two curves



• fixed pts

At pt P_1 in xr-plane we think it is a subcritical pitchfork

P_2 & P_3 we think they are simultaneous saddle-node bifurcations (sub-crit)

Saddle node

A, B $\neq 0$, but what about C?

$$\dot{x} = Ar + Bx^2 + Cxr$$

$(x, r) = 0$ BIF PT.

$$= Ar + B\left(x + \frac{Cr}{2B}\right)^2 - \frac{C^2r^2}{4B^2}$$

$$= A\left(r - \frac{C^2r^2}{4AB^2}\right) + B\left(x + \frac{Cr}{2B}\right)^2$$

$$\left.\begin{aligned} y &= x + \frac{Cr}{2B} \\ \mu &= r - \frac{C^2r^2}{4AB^2} \end{aligned}\right\} \quad \dot{y} = \dot{x} = A\mu + By^2.$$

$$\left.\frac{d\mu}{dr}\right|_{r=0} = 1$$

$r = \mu^2$. X $\quad \mu = r^2$ x

So value of C is not important if A, B $\neq 0$

$\dot{x} = x^4 + \mu x^2 + \nu x$

S-N.

$\dot{x} = 0, 1, 2.$ FPs.

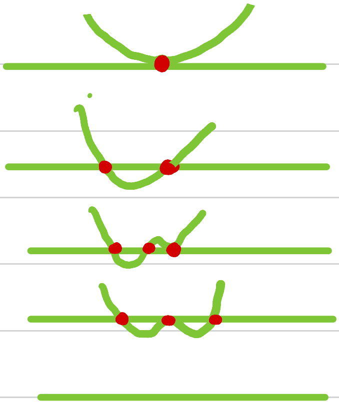
$\mu = \nu = 0$

$\nu > 0, \mu = 0$

$\nu = 0 \quad \mu > 0$

$\nu = 0 \quad \mu < 0$

$\nu > 0 \quad \mu < 0$



?

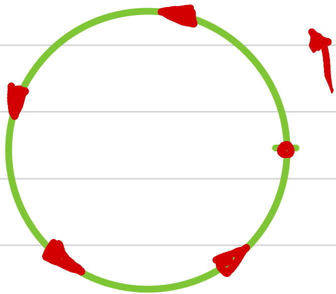
This meant to show there are bifurcation diagram with more than 1-parameter !!

3. Dynamics on the Circle S^1

S^1

$\dot{\theta} = 1$ ✓

$\dot{\theta} = \theta^2$ ✗



$\dot{\theta} = 1 - \cos\theta$

$\theta = 0$ FPs

$\dot{\theta} = 0$

i.e. $1 - \cos\theta = 0$

$\theta = 0$ FP

note $\dot{\theta} > 0$, $\theta \neq 0$

$\dot{\theta} = f(\theta)$

needs $f(\theta + 2\pi) = f(\theta)$