WEEK 11 NOTES

1. Further properties of the heat kernel and its application to solve the heat equations

Recall last week that we obtained a special solution to the heat equation called the heat kernel

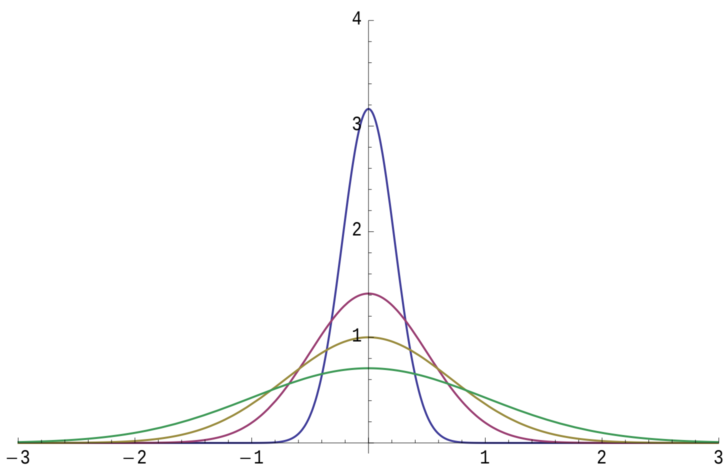
$$K(x,t) = \frac{e^{-\frac{x^2}{4\varkappa t}}}{\sqrt{4\varkappa \pi t}}.$$

1.1. **Dirac's delta function.**

In order to better understand the properties of the heat Kernel at $t = 0$ consider the sequence of functions

$$\{f_\lambda(x)\} = \left\{ \frac{e^{-\frac{x^2}{\lambda^2}}}{\lambda\sqrt{\pi}} \right\}, \qquad \lambda \in \mathbb{R}^+.$$

Graphs of the functions f_λ for various values of λ can be seen in the figure below:



Observe that as $\lambda \to 0$, the Gaussian bells become increasingly peaked. One can then check that:

(i) if $x \neq 0$ then $f_\lambda(x) \to 0$ as $\lambda \to 0$;
(ii) if $x = 0$ then $f_\lambda(0) \to \infty$ as $\lambda \to 0$;
(iii) moreover, one has that

$$\int_{-\infty}^{\infty} f_\lambda(x)dx = 1$$

for all λ so that, in particular, one has

$$\lim_{\lambda\to 0} \int_{-\infty}^{\infty} f_\lambda(x)dx = 1.$$

The limit of the family $\{f_\lambda\}$ is not a proper function. However, one can *formally* write

$$\delta(x) \equiv \lim_{\lambda\to 0} f_\lambda(x) = \lim_{\lambda\to 0} \frac{e^{-x^2/\lambda^2}}{\sqrt{\pi}}.$$

This is the so-called *Dirac's delta "function"*.

Note. *There is a branch of mathematics known as* distribution theory *aimed at making sense of objects like Dirac's delta.*

Definition 1.1. Dirac's delta, δ, is defined by the conditions:
(i) $\delta(x) = 0$ for $x \neq 0$;
(ii) $\delta(0) = \infty$;
(iii) for any $a < 0 < b$ one has

$$\int_a^b \delta(x) = 1.$$

Note. *From the previous discussion it follows that*

$$K(x,0) = \delta(x).$$

In terms of diffusion processes, $\delta(x)$ describes an infinitesimally small "drop" of ink concentrated at the origin. This "drop" then spreads with time.

1.2. **The general solution to the heat equation on the real line.** The heat kernel is the basic building block to obtain the general solution to the heat equation on the real line.

We begin by observing the following property:

Lemma 1.2. *If $U(x,t)$ is a solution to*

$$U_t = \varkappa U_{xx}$$

then

$$V(x,t) \equiv \int_{-\infty}^{\infty} U(x-y,t)g(y)dy \tag{1.1}$$

is also a solution for any function g —as long as the integral converges.

Proof. This follows by direct computation:

$$V_t(x,t) = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} U(x-y,t)g(y)dy = \int_{-\infty}^{\infty} U_t(x-y,t)g(y)dy,$$

$$V_{xx}(x,t) = \frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty} U(x-y,t)g(y)dy = \int_{-\infty}^{\infty} U_{xx}(x-y,t)g(y)dy.$$

Hence,

$$V_t(x,t) - \varkappa V_{xx}(x,t) = \int_{-\infty}^{\infty} \Big(U_t(x-y,t) - \varkappa U_{xx}(x-y,t)\Big)g(y)dy = 0.$$

□

Note. *The operation given by* (1.1) *is called the* convolution *of U and g. This is sometimes denoted as*

$$V(x,t) = (U * g)(x,t).$$

Now, consider the problem

$$U_t = \varkappa U_{xx}, \qquad x \in \mathbb{R}, \quad t > 0, \tag{1.2}$$

$$U(x,0) = f(x). \tag{1.3}$$

Claim: the (unique) solution to (1.2)-(1.3) is given by

$$U(x,t) = \int_{-\infty}^{\infty} K(x-y,t) f(y) dy,$$

with K denoting the heat kernel. Or, more explicitly,

$$U(x,t) = \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{4\varkappa t}}}{\sqrt{4\varkappa\pi t}} f(y) dy. \tag{1.4}$$

The latter is known as the *Fourier-Poisson formula*.

As a consequence of Lemma 1.2, and given that $K(x,t)$ satisfies the heat equation, then $U(x,t)$ as defined by (1.4) is a solution to the heat equation.

Note. *To fully address the claim it is only necessary to verify that* $U(x,0) = f(x)$.

1.2.1. *Some auxiliary calculations.* In the following it will be convenient to consider the function

$$Q(x,t) \equiv \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4\varkappa t}} e^{-s^2} ds, \qquad t > 0. \tag{1.5}$$

Observe that

$$\begin{aligned} Q_x(x,t) &= \frac{1}{\sqrt{\pi}} \frac{d}{dx} \left(\frac{x}{\sqrt{4\varkappa t}} \right) e^{-\frac{x^2}{4\varkappa t}} \\ &= \frac{e^{-\frac{x^2}{4\varkappa t}}}{\sqrt{4\pi\varkappa t}} = K(x,t). \end{aligned}$$

Thus, $Q(x,t)$ is the antiderivative (with respect to x) of $K(x,t)$.

Next, we consider the limit of $Q(x,t)$ as $t \to 0^+$. There are 2 cases:

(i) $x > 0$. Here we have

$$\lim_{t\to 0^+} Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-s^2} ds = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} = 1.$$

(ii) $x < 0$. Here one has

$$\lim_{t\to 0^+} Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-s^2} ds = \frac{1}{2} - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{0} e^{-s^2} ds = \frac{1}{2} - \frac{1}{2} = 0.$$

Hence, one concludes that

$$\lim_{t\to 0^+} Q(x,t) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases} \equiv H(x).$$

The function H defined above is called *Heaviside's step function*.

Note. *As Q is the antiderivative of K it follows from the above discussion that*

$$H'(x) = \delta(x).$$

That is, Dirac's delta is the derivative of Heaviside's step function.

1.2.2. *Concluding the main computation.* Using the properties of Q as discussed in the previous subsection one has that

$$\begin{aligned}U(x,t) &= \int_{-\infty}^{\infty} K(x-y,t)f(y)dy \\ &= \int_{-\infty}^{\infty} Q_x(x-y,t)f(y)dy \\ &= -\int_{-\infty}^{\infty} Q_y(x-y,t)f(y)dy \\ &= -Q(x-y,t)f(y)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} Q(x-y,t)f'(y)dy,\end{aligned}$$

where in the third line one makes use of the chain rule to change the x-derivative to a y-derivative and in the fourth line one employs integration by parts to pass the derivative from Q to f. Now, as $K(x-y,t)$ decays very fast to 0 as $|x-y| \to \infty$ it follows that

$$-Q(x-y,t)f(y)\Big|_{-\infty}^{\infty} = 0.$$

Hence,

$$U(x,t) = \int_{-\infty}^{\infty} Q(x-y,t)f'(y)dy.$$

We make use of this expression to compute the limit $t \to 0^+$:

$$\begin{aligned}U(x,0^+) &= \int_{-\infty}^{\infty} Q(x-y,0^+)f'(y)dy = \int_{-\infty}^{\infty} H(x-y)f'(y)dy \\ &= \int_{-\infty}^{x} f'(y)dy = f(y)\Big|_{-\infty}^{x} = f(x),\end{aligned}$$

where in the last line it has been assumed that $f(x) \to 0$ as $x \to -\infty$.

We summarise the previous discussion in the following:

Proposition 1.3. *For $t > 0$, the* ***Fourier-Poisson formula***

$$\boxed{U(x,t) = \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{4\varkappa t}}}{\sqrt{4\varkappa\pi t}} f(y)dy}$$

gives the (unique) solution to

$$\begin{aligned}&U_t(x,t) = \varkappa U_{xx}(x,t), \qquad x \in \mathbb{R}, \quad t > 0, \\ &U(x,0) = f(x).\end{aligned}$$

1.3. **Some examples.** In this section we discuss some examples of computation involving the Fourier-Poisson formula.

Example 1.4. Analyse the behaviour of the solution $U(x,t)$ given by the Fourier-Poisson formula in the case

$$f(x) = H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}.$$

In this case one has

$$U(x,t) = \frac{1}{\sqrt{4\pi\varkappa t}}\int_{-\infty}^{\infty} e^{-(x-y)^2/4\varkappa t} H(y)dy$$
$$= \frac{1}{\sqrt{4\pi\varkappa t}}\int_{0}^{\infty} e^{-(x-y)^2/4\varkappa t} dy$$

as $H(x) \neq 0$ only for $x > 0$. Letting now

$$s = \frac{x-y}{\sqrt{4\varkappa t}} \implies dy = -\sqrt{4\varkappa t}ds,$$

one finds that

$$U(x,t) = -\frac{\sqrt{4\varkappa t}}{\sqrt{4\pi\varkappa t}}\int_{x/\sqrt{4\varkappa t}}^{-\infty} e^{-s^2}ds$$
$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{x/\sqrt{4\varkappa t}} e^{-s^2}ds$$
$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{0} e^{-s^2}ds + \frac{1}{\sqrt{\pi}}\int_{0}^{x/\sqrt{4\varkappa t}} e^{-s^2}ds$$
$$= \frac{1}{2} + \frac{1}{\sqrt{\pi}}\int_{0}^{x/\sqrt{4\varkappa t}} e^{-s^2}ds.$$

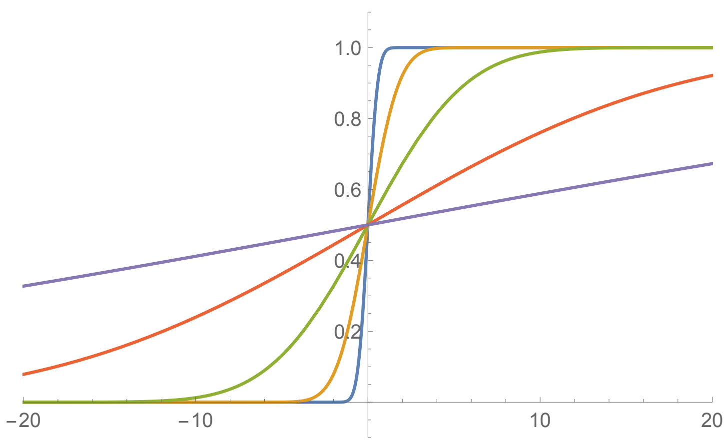
Thus, observe that, in fact

$$U(x,t) = Q(x,t).$$

We now investigate the behaviour of $U(x,t)$ for fixed x as $t \to \infty$:

$$\lim_{t\to\infty} U(x,t) = \frac{1}{2} + \lim_{t\to\infty}\frac{1}{\sqrt{\pi}}\int_{0}^{x/\sqrt{4\varkappa t}} e^{-s^2}ds = \frac{1}{2} + \frac{1}{\sqrt{\pi}}\int_{0}^{0} e^{-s^2}ds = \frac{1}{2},$$

as $x/\sqrt{4\varkappa t} \to 0$ as $t \to \infty$.



Example 1.5. Evaluate the Fourier-Poisson formula in the case

$$f(x) = e^{-x}.$$

Substituting the above expression in the formula one obtains

$$U(x,t) = \frac{1}{\sqrt{4\varkappa\pi t}}\int_{-\infty}^{\infty} e^{-(x-y)^2/4\varkappa t} e^{-y}dy.$$

The exponent in the integral can be rearranged as

$$-\frac{(x-y)^2}{4\varkappa t} - y = -\frac{x^2 - 2xy + y^2 + 4\varkappa t y}{4\varkappa t}$$
$$= -\frac{(y + 2\varkappa t - x)^2}{4\varkappa t} + \varkappa t - x.$$

Hence,

$$U(x,t) = \frac{1}{\sqrt{4\varkappa\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(y+2\varkappa t-x)^2}{4\varkappa t} + \varkappa t - x} dy$$
$$= \frac{e^{\varkappa t - x}}{\sqrt{4\varkappa\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(y+2\varkappa t-x)^2}{4\varkappa t}} dy.$$

Letting

$$s = \frac{y + 2\varkappa t - x}{\sqrt{4\varkappa t}} \Longrightarrow ds = \frac{dy}{\sqrt{4\varkappa t}},$$

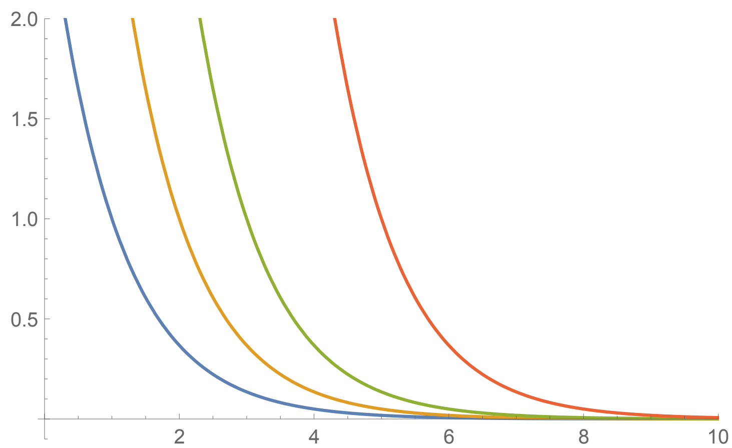
it follows then that

$$U(x,t) = \frac{e^{\varkappa t - x}}{\sqrt{4\varkappa\pi t}} \int_{-\infty}^{\infty} e^{-s^2} \sqrt{4\varkappa t} ds$$
$$= \frac{e^{\varkappa t - x}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-s^2} ds = e^{\varkappa t - x}.$$

Observe, in particular, that

$$U(x,t) \to \infty, \qquad \text{as} \quad t \to \infty.$$

Thus, the solution does not decay but grows at every point x. Plots of this solution for various values of t are shown below.



We conclude the list of examples with one particular solution to the heat equation which evidences an important property of the heat equation:

Example 1.6. Compute the solution to the heat equation on the real line if the initial condition is given by

$$f(x) = \frac{e^{x^2/4\varkappa}}{\sqrt{4\pi\varkappa}}.$$

In this case substitution of the initial condition into the Fourier-Poisson formula gives

$$\begin{aligned} U(x,t) &= \frac{1}{\sqrt{4\pi\varkappa t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\varkappa t}}\frac{e^{\frac{y^2}{4\varkappa}}}{\sqrt{4\pi\varkappa}}dy \\ &= \frac{1}{4\pi\varkappa\sqrt{t}}\int_{-\infty}^{\infty} e^{\frac{y^2}{4\varkappa}-\frac{(x-y)^2}{4\varkappa t}}dy. \end{aligned}$$

The exponent in the integral can be manipulated by completing squares to get

$$\begin{aligned} \frac{y^2}{4\varkappa}-\frac{(x-y)^2}{4\varkappa t} &= \frac{1}{4\varkappa}\left(y^2-\frac{x^2}{t}-\frac{y^2}{t}+\frac{2xy}{t}\right) \\ &= \frac{1}{4\varkappa}\left(\left(1-\frac{1}{t}\right)y^2+\frac{2xy}{t}-\frac{x^2}{t}\right) \\ &= \frac{1}{4\varkappa}\left(\left(\frac{t-1}{t}\right)\left(y^2+\frac{2xy}{t-1}+\frac{x^2}{(t-1)^2}\right)-\frac{x^2}{t}-\frac{x^2}{t(t-1)}\right) \\ &= \frac{1}{4\varkappa}\left(\left(\frac{t-1}{t}\right)\left(y+\frac{x}{t-1}\right)^2-\frac{x^2}{t-1}\right). \end{aligned}$$

Hence,

$$\begin{aligned} U(x,t) &= \frac{1}{4\pi\varkappa\sqrt{t}}\int_{-\infty}^{\infty} e^{\frac{1}{4\varkappa}\left(\left(\frac{t-1}{t}\right)\left(y+\frac{x}{t-1}\right)^2-\frac{x^2}{t-1}\right)}dy \\ &= \frac{e^{-\frac{x^2}{4\varkappa(t-1)}}}{4\pi\varkappa\sqrt{t}}\int_{-\infty}^{\infty} e^{-\frac{1}{4\varkappa}\left(\frac{1-t}{t}\right)\left(y+\frac{x}{t-1}\right)^2}dy. \end{aligned}$$

Finally, letting

$$s=\sqrt{\frac{1-t}{4\varkappa t}}\left(y+\frac{x}{t-1}\right) \Longrightarrow ds=\sqrt{\frac{1-t}{4\varkappa t}}dy,$$

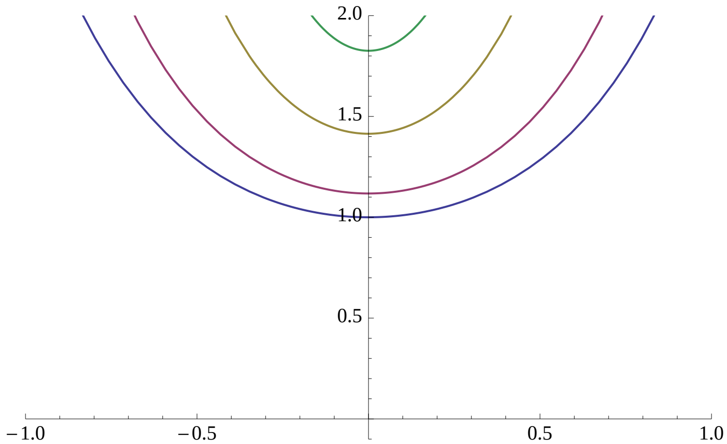
one concludes that

$$\begin{aligned} U(x,t) &= \frac{e^{-\frac{x^2}{4\varkappa(t-1)}}}{4\pi\varkappa\sqrt{t}}\sqrt{\frac{4\varkappa t}{1-t}}\int_{-\infty}^{\infty} e^{-s^2}ds \\ &= \frac{e^{\frac{x^2}{4\varkappa(1-t)}}}{\pi\sqrt{4\varkappa(1-t)}}\sqrt{\pi} \\ &= \frac{e^{\frac{x^2}{4\varkappa(1-t)}}}{\sqrt{4\varkappa\pi(1-t)}}. \end{aligned}$$

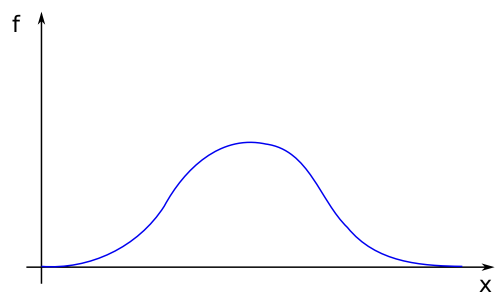
Observe that

$$U(x,t)\longrightarrow\infty \qquad \text{as} \quad t\to 1.$$

That is, the solution becomes singular in a finite amount of time! A plot of the solutions for various values of t is shown below:



Example 1.7. In this example we suppose the initial date f is a bump function so that $f = 0$ for $|x| \geq R_0$ and $|f(x)| \leq C_0$. See for example the of a bump function graphed below.



Then by the Fourier-Poisson formula, we get

$$\begin{aligned} U(x,t) =& \frac{1}{\sqrt{4\pi\varkappa t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\varkappa t}} f(y)dy \\ =& \frac{1}{\sqrt{4\pi\varkappa t}} \int_{-R_0}^{R_0} e^{-\frac{(x-y)^2}{4\varkappa t}} f(y)dy \\ \leq& \frac{1}{\sqrt{4\pi\varkappa t}} \cdot 2R_0 \cdot C_0 \\ \to& 0, \text{ as } t \to \infty. \end{aligned}$$

Here in the second line we used that $f = 0$ for $x \geq R_0$ or $x \leq -R_0$.

We see that after a very long time, the solution to the heat equation tend to zero!