MTH6151: Partial Diff. Equations.
Solutions final exam 2018/2019.

Question 1.

a) The method of characteristics transforms the pde

$$a(x,y)\,u_x + b(x,y)\,u_y = c(x,y)\,u + d(x,y)$$

into the problem of solving the ordinary differential equations

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{b(x,y(x))}{a(x,y(x))} & (*_1) \\[2ex] \dfrac{du}{dx}(x,y(x)) = \dfrac{c(x,y(x))}{a(x,y(x))}\,u(x,y(x)) + \dfrac{d(x,y(x))}{a(x,y(x))} & (*_2) \end{cases}$$

Equation $(*_1)$ gives the characteristic curves while $(*_2)$ allows to find u along the charact. curves (transport equation).
[Book work] [4 marks]

b) i) $u_x + \tan x\, u_{yy} - u = \cos y$

Linear second order, inhomogeneous equation. [2 marks]

ii) $5uu_{tt} - u^2 u_x = 0$.

Nonlinear second order, inhomogeneous. [2 marks]

[Bookwork].

c) Solve $u_x - 2u_t = 0$.
$u(0,t) = \cos t$

The characteristic curves satisfy $\frac{dt}{dx} = -2$.

$\Rightarrow t = -2x + C \iff C = t + 2x$. [2 marks]

Moreover, from general theory $\frac{du}{dx} = 0$ along a characteristic.

$\therefore \; u(x,t) = f(C) = f(2x+t)$. [1 mark]

In particular, $u(0,t) = f(C)$ as $C = t$ if $x = 0$,
but $u(0,t) = \cos t = \cos C \Rightarrow f(C) = \cos C$.

Hence, the required solution is $\boxed{u(x,t) = \cos(2x+t)}$. [2 marks]
[Similar to CW/ lectures]

d) Find the general solution to

$$u_t + x\,u_x = \sin t$$

The eqn. for the characteristics is $\frac{dt}{dx} = \frac{1}{x} \Rightarrow t(x) = \ln x + C$. [2 marks]

Thus, $\frac{du}{dx} = \frac{\sin t}{x}$ (from general theory)

Eliminating t one gets $\frac{du}{dx} = \frac{\sin \ln(Cx)}{x}$,

but $\int \frac{\sin \ln(Cx)}{x}\, dx = \int \sin z \, dz = -\cos z = -\cos \ln Cx$.

Hence, $u = -\cos \ln Cx + f(C)$.

Now, as $C = e^t/x$ one concludes that

$$u(x,t) = -\cos t + f\left(\frac{e^t}{x}\right).$$ [5 marks]

[CW / lectures]

Question 2.

a) Classify

$$\begin{cases} b^2 - ac > 0 & \text{hyperbolic} \\ b^2 - ac = 0 & \text{parabolic} \\ b^2 - ac < 0 & \text{elliptic} \end{cases}$$

i) $2u_{xx} - 4u_{xy} - 6u_{yy} + u_x = 0$

Here $a = 2$, $b = -2$, $c = -6 \Rightarrow 4 + 2 \cdot 6 = 16 > 0$

$\therefore$ Hyperbolic equation. [2 marks]

ii) $u_{xx} + 2u_{xy} + 17u_{yy} = 0$.

$a = 1$, $b = 1$, $c = 17$

$\Rightarrow \quad 1 - 1 \cdot 17 = -16 < 0$

$\therefore$ Elliptic equation. [2 marks]

[Bookwork]

b) Given $f(x)$ differentiable

i) Show that $u(x,t) = f(x+ct)$ solves

$$u_t - c u_x = 0.$$

Using the chain rule $u_x = \frac{\partial u}{\partial x} = f'(x+ct)$ [1 marks]

$$u_t = \frac{\partial u}{\partial t} = f'(x+ct)\frac{d(ct)}{dt}$$

$$= cf'(x+ct).$$ [1 marks]

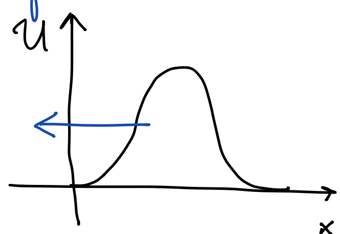
Thus, readily

$$u_t - cu_x = c\left(f'(x+ct) - f'(x+ct)\right) = 0. \quad [1\text{ mark}]$$

[Similar to CW / lectures]

ii) Assume f has the form

Let the maximum of f be at $x = x_*$. Then for $t>0$ one will have that $x_* = x+ct \Rightarrow x = x_* - ct$. Thus, for $t>0$ the initial profile moves to the left an amount ct keeping its shape:



[3 marks]

iii) If $u(x,0)=0$ then $u(x,t)=0$ for all t. So if there is no initial profile then it cannot arise at latter times.

[Similar to CW / lectures] [2 marks].

Question 3.

a) Given

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

The term $\frac{1}{2}\left(f(x+ct)+f(x-ct)\right)$ gives the average of f at the points $(x-ct)$ and $(x+ct)$.

On the other hand,

$$\frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$ gives the average of g on the interval $[x-ct, x+ct]$.

[4 marks]

[Bookwork].

b) If $u(x,t)$ is a solution to the wave equation then $V(x,t) \equiv u(\alpha x, \alpha t)$ is also a solution.

Let $\begin{cases} v = \alpha x \\ w = \alpha t \end{cases} \Longrightarrow V(x,t) = u(v,w).$

Hence $\begin{cases} \dfrac{\partial}{\partial x} = \dfrac{dv}{dx}\dfrac{\partial}{\partial v} = \alpha \dfrac{\partial}{\partial v} \\ \dfrac{\partial}{\partial t} = \dfrac{dw}{dt} \cdot \dfrac{\partial}{\partial w} = \alpha \dfrac{\partial}{\partial w} \end{cases}$ [2 marks]

$$\Rightarrow \frac{\partial^2}{\partial x^2} = \alpha^2 \frac{\partial^2}{\partial v^2} \quad ; \quad \frac{\partial^2}{\partial t^2} = \alpha \frac{\partial^2}{\partial w^2} .$$ [2 marks]

Thus,

$$\frac{\partial^2 V}{\partial t^2} - \frac{c^2 \partial^2 V}{\partial x^2} = \alpha^2 \left(\frac{\partial^2}{\partial w^2} u(v,w) - c^2 \frac{\partial^2}{\partial v^2} u(v,w) \right)$$

$= 0$ as u is a solution to the wave eqn.

[Similar to HW/ lectures] [2 marks]

c) Find the solution to the problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \ , \ x \in \mathbb{R} \\ u(x,0) = \dfrac{1}{1+x^2} , \\ u_t(x,0) = 0 . \end{cases}$$

One can use D'Alembert's formula to directly write the solution.

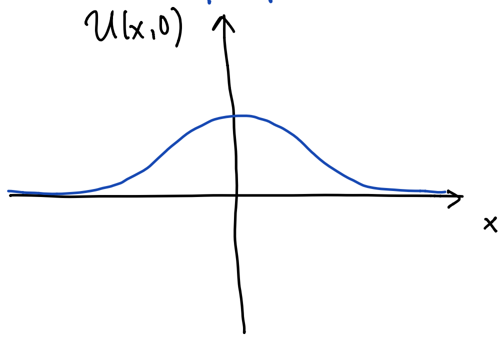
Setting $f(x) = \dfrac{1}{1+x^2}$

one obtains

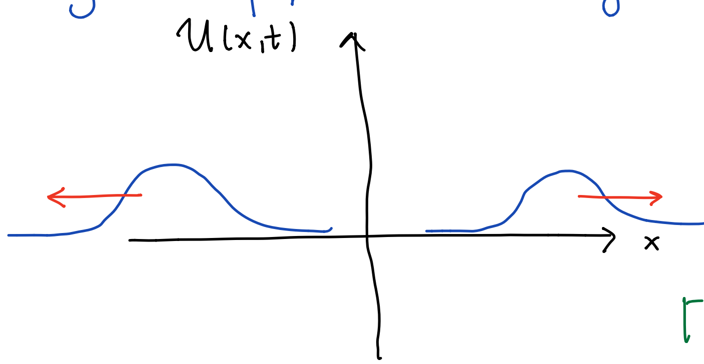
$$u(x,t) = \frac{1}{2}\left(\frac{1}{1+(x+ct)^2} + \frac{1}{1+(x-ct)^2}\right).$$

[3 marks]

Initially one has the profile



For $t>0$ the profile splits into two bits of half the original size — one moving to the left, the other to the right.



[3 marks]

[Similar to cw/lectures]

d) The wave equation has a finite speed of propagation (c) while the heat equation has infinite speed of propagation.

[4 marks]

[Bookwork]

Question 4.

Consider
$$\begin{cases} u_{xx} + u_{yy} = 0 & (x,y) \in \Omega = \{0<x<a,\ 0<y<b\}. \\ u(x,0) = 0,\ u(x,b) = f(x), & \\ u(0,y) = 0,\ u(a,y) = 0. & \end{cases}$$

a) Substituting $u(x,y) = X(x)Y(y)$ into $u_{xx} + u_{yy} = 0$ one readily obtains

$$X''(x)Y(y) + X(x)Y''(y) = 0.$$

$$\Rightarrow \quad \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = k \quad \text{a constant} \qquad [2\text{ marks}]$$

as the LHS depends on x only and the RHS depends on y only.

Hence,
$$\begin{cases} X'' = kX \\ Y'' = -kY \end{cases} \qquad [2\text{ marks}]$$

As $u(0,y) = u(a,y) = 0$, so that

$$X(0)Y(y) = X(a)Y(y) = 0 \quad \text{as } Y(y) \text{ not identically vanishing.}$$

$$\therefore\ X(0) = X(a) = 0.$$

Similarly, $X(x)Y(0) = 0$ so that $Y(0) = 0$. [2 marks]

b) Show that $k<0$ if $X \neq 0$.

As $X(0) = X(a) = 0$, then we expect periodic solutions so that, accordingly, $k<0$. To prove this consider

$$X'' - kX = 0 \Rightarrow \int_0^a X(X'' - kX)\,dx = 0.$$ [2 marks]

$$\Longrightarrow \int_0^a XX''\,dx - k\int_0^a X^2\,dx$$

$$= XX'\Big|_0^a - \int_0^a X'^2\,dx - k\int_0^a X^2\,dx$$

int. by parts

$$= -\int_0^a X'^2\,dx - k\int_0^a X^2\,dx = 0$$ [2 marks]

$$\Longrightarrow -k\int_0^a X^2\,dx = \int_0^a X'^2\,dx > 0 \quad \text{for } X \neq 0.$$

[2 marks]

Thus, $k<0$ ■

c) As $k<0$, let $k = -\mu^2$, so that

$$\begin{cases} X(x) = A\sin\mu x + B\cos\mu x, & \text{[2 marks]} \\ Y(y) = C\sinh\mu x + D\cosh\mu y. & \text{[2 marks]} \end{cases}$$

d) Given $X(0) = X(a) = 0$, it follows from

$$X(x) = A\sin\mu x + B\cos\mu x$$

that

$$X(0) = A\sin 0 + B\cos 0 = B = 0. \qquad \text{[2 marks]}$$

$$\Rightarrow X(a) = A\sin\mu a = 0 \quad \Rightarrow \mu a = n\pi, \quad n = 1, 2, 3, \ldots$$

$$\therefore X(x) = \sin\left(\frac{n\pi x}{a}\right).$$

Also, as $Y(0) = 0$, then from

$$Y(y) = C\sinh\mu y + D\cosh\mu y$$

one gets

$$Y(0) = C\sinh 0 + D\cosh 0 = D = 0.$$

$$\therefore Y(y) = \sinh\left(\frac{n\pi y}{a}\right). \qquad \text{[2 marks]}$$

e) Collecting the previous calculations one finds the following family of solutions to the Laplace equation:

$$U_n(x,y) = \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right) \qquad n = 1, 2, 3, \ldots \quad \text{[2 marks]}$$

As the Laplace eqn is linear, the principle of superposition applies and the most general solution is given by

$$U(x,y) = \sum_{n=1}^{\infty} a_n U_n(x,y)$$

$$= \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right) \qquad [2 \text{ marks}]$$

with a_n constants.

f) Evaluating at $y=b$ one has

$$U(x,b) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi b}{a}\right)$$

$$= \sin\left(\frac{5\pi x}{a}\right) + 2\sin\left(\frac{6\pi x}{a}\right).$$

Comparing coefficients one finds [2 marks]

$$\sinh\left(\frac{5\pi b}{a}\right) a_5 = 1, \quad \sinh\left(\frac{6\pi b}{a}\right) a_6 = 2.$$

$$\therefore U(x,y) = \frac{1}{\sinh(5\pi b/a)} \sin\left(\frac{5\pi x}{a}\right) \sinh\left(\frac{5\pi x}{a}\right)$$

$$+ \frac{2}{\sinh(6\pi b/a)} \sin\left(\frac{6\pi x}{a}\right) \sinh\left(\frac{6\pi x}{a}\right). \qquad [2 \text{ marks}]$$

[Similar to CW problems]

Question 5.

a) The Fourier – Poisson formula

$$u(x,t) = \int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/4\varkappa t}}{\sqrt{4\pi\varkappa t}}\, f(y)\,dy$$

gives the (unique) solution to the initial value problem to the heat equation on the real line. [Bookwork] [2 marks]

b) Show that

$$u(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4\varkappa t}} e^{-s^2}\,ds$$

is a solution to the heat equation. Find the value of $u(x,0^+)$, $x>0$.

Using the Fundamental Theorem of Calculus one gets

$$u_x(x,t) = \frac{1}{\sqrt{\pi}} e^{-x^2/4\varkappa t} \frac{d}{dx}\left(\frac{x}{\sqrt{4\varkappa t}}\right)$$

$$= \frac{1}{\sqrt{\pi}} e^{-x^2/4\varkappa t} \frac{1}{\sqrt{4\varkappa t}} = \frac{1}{\sqrt{4\pi\varkappa t}} e^{-x^2/4\varkappa t}.$$

$$\Rightarrow u_{xx}(x,t) = -\frac{2x}{4\varkappa t} \cdot \frac{1}{\sqrt{4\pi\varkappa t}} \frac{e^{-x^2/4\varkappa t}}{\sqrt{\pi}} = -\frac{1}{2} \frac{x}{\varkappa t\sqrt{4\pi\varkappa t}} \cdot \frac{e^{-x^2/4\varkappa t}}{\sqrt{\pi}}$$

[2 marks]

Also, from the chain rule,

$$u_t(x,t) = \frac{1}{\sqrt{\pi}} e^{-x^2/4\varkappa t} \frac{d}{dt}\left(\frac{x}{\sqrt{4\varkappa t}}\right) = -\frac{1}{2}\frac{1}{\sqrt{\pi}}\frac{x}{\sqrt{4\pi\varkappa}} \cdot \frac{1}{t\sqrt{t}} e^{-x^2/4\varkappa t}.$$

[2 marks]

$$\Longrightarrow u_t - \varkappa u_{xx} = 0.$$

Now, taking the limit $t \to 0$ one has that

$$\lim_{t\to 0} u(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-s^2} ds = \frac{1}{2} + \frac{1}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} = 1.$$

Appendix.

[Unseen] [2 marks]

c) Maximum principle for the heat equation.

Given a solution $u(x,t)$ to the heat equation on the "rectangle"

$$\Omega = \{(x,t) \mid 0 < x < L,\ 0 < t < T\},$$

the maximum of $u(x,t)$ on Ω is attained at either the initial surface $t=0$ or at one of the boundaries $x=0$ or $x=L$.

[Bookwork] [4 marks]

d) Consider $u(x,t) = 1 - x^2 - 2\varkappa t$, solution to the heat equation. Find its maxima/minima in

$$\Omega = \{0 < x < 1,\ 0 < t < T\}.$$

From the Principle of the Maximum one has that the maxima/minima can only occur at

$$\begin{cases} t = 0 & , \quad 0 \leq x \leq 1 \\ x = 0 & , \quad 0 \leq t \leq T \\ x = 1 & , \quad 0 \leq t \leq T \end{cases}$$

[1 mark]

- Now $u(x,0) = 1 - x^2$, $u'(x,0) = -2x < 0$ so that there is an extremum at $x = 0$. As $u''(x,0) = -2 < 0$ one has a local maximum of $u(x,0)$, $0 \leq x \leq 1$.

  $\therefore$ The maximum/minimum of $u(x,0)$ for $0 \leq x \leq 1$ occur, respectively at $x = 0$, $x = 1$.

[2 marks]

- Now, look at the left boundary. In this case

$$u(0,t) = 1 - 2\varkappa t \quad , \quad 0 \leq t \leq T.$$

Thus $u(0,0) = 1$ (maximum on the side), $u(0,T) = 1 - 2\varkappa T < 1$ (minimum on the side).

[2 marks]

On the right boundary one has that $u(1,t) = -2xt < 0$.

Thus, $u(1,0) = 0$ (maximum on the side) $\quad u(1,T) = -2xT < 0$ (minimum on the side)

Collecting the above, the minimum of $u(x,t)$ occurs at $(1,T)$ while the maximum is attained at $(0,0)$. [1 mark]

[Similar to CW/lectures].