MTH6151: Partial diff. eqns. Solutions to the January 2021 exam.

Question 1.

a) (i) What happens when the characteristic curves of a first order linear pde do not cover the whole plane?

↳ If there exists a point $(x_*, y_*) \in \mathbb{R}$ which is not covered by a characteristic curve, then the solution $u(x,y)$ is not defined at the point. [3 marks] [Book work].

(ii) What happens when two (or more) characteristics intersect?

↳ The solution of the transport equation from which one computes the solution to the pde is not defined uniquely. Hence, the solution $u(x,y)$ to the pde breaks down at the intersection of the solutions. [3 marks] [Bookwork]

b) State which of the following eqns. are linear or non-linear.

i) $u_x + e^y u_y = 0$. (linear) [1 mark]

ii) $$\frac{u_x}{1+u_x^2} + \frac{u_y}{1+u_y^2} = 0.$$ (non-linear). [1 mark]

[Bookwork].

(c) Find the solution to

$$u_x - 2u_t = 0, \quad u(0,t) = \tanh t.$$

↳ This is a first order pde with constant coeffs. The general solution is given by

$$u(x,t) = f(bx - at) \quad \text{where } f \text{ is an arbitrary function.}$$

Here $a=1$, $b=2$. Thus,

$$u(x,t) = f(-2x-t) = f(2x+t)$$ (redefining f). [2 marks]

Now, $u(0,t) = f(t) = \tanh t$ so that the particular solution we look for is [1 mark]

$$u(x,t) = \tanh(2x+t).$$

[Similar to CW]

ii) In the previous problem the information is provided on the vertical axis. Thus, this is a boundary value problem.

[1 mark]
[Book work]

d) $$\begin{cases} N_t + N_x = -e^x N \\ N(x,0) = f(x) \\ N(0,t) = e^{-t} \end{cases}$$

↳ In this population model the mortality rate $\mu(x,t) = e^x$ increases for older age groups. There are, however, no seasonal variation on the mortality rate. The initial population distribution is $f(x)$. The boundary condition e^{-t} gives the number of births at time t. It decreases quickly with time. [3 marks] [Similar to discussion in class]

e) i) Find the solution to

$$xU_x + yU_y = kU + 1, \quad k \text{ constant.}$$

↪ The eqn. for the characteristic curves is:

$$\frac{dy}{dx} = \frac{y}{x}$$ [2 marks]

The solution is given by $y(x) = Cx$ (lines) C a constant.

Now, the transport eqn along the characteristics is

$$\frac{dU(x, y(x))}{dx} = \frac{k}{x} U(x, y(x)) + \frac{1}{x}.$$ [1 mark]

Write for simplicity as

$$U' - \frac{k}{x} U = \frac{1}{x} \rightsquigarrow \text{integrating factor } e^{-k\int \frac{dx}{x}} = e^{-k \ln x} = x^{-k}.$$

$$\Rightarrow \underbrace{x^{-k} U' - k x^{-k-1} U}_{(x^{-k} U)'} = x^{-k-1}$$

an arb. funct. of C.

So, integrating: $$x^{-k} U = \int x^{-k-1} dx + f(C)$$

$$= -\frac{1}{k} x^{-k} + f(C)$$

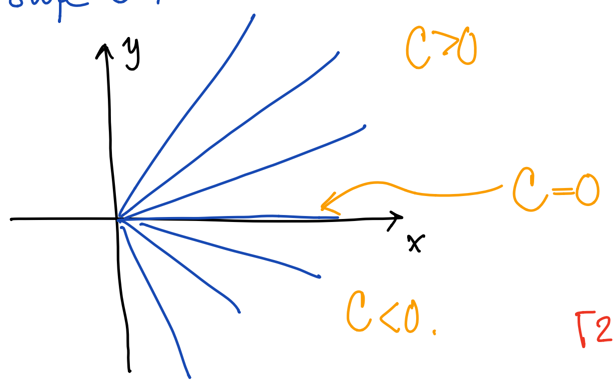
$$\Rightarrow U(x, y(x)) = -\frac{1}{k} + x^k . f(C).$$

[3 marks]

But $C = y/x$

$$\therefore U(x,y) = -\frac{1}{k} + x^k f(y/x).$$

ii) The characteristics are lines $y = Cx$ passing through the origin with slope C:



[2 marks].

[Similar to examples in CW].

Question 2.

a) When is $a\,u_{xx} - a\,u_{xy} - c\,u_x + d\,u_y = f$ elliptic?

↳ The discriminant reads $(a/2)^2 \geq 0$. Thus, the eqn. can never be elliptic!

[2 marks]
[similar to CW]

b) When is $a\,u_{xy} + b\,u_x + c\,u_y + u = 0$ parabolic?

↳ The discriminant is, again, $(a/2)^2 > 0$. So the eqn. can never be parabolic.

[2 marks]
[similar to CW]

c) Conserved quantity for a solution to the wave eqn.

↳ A conserved quantity for a soln. to the wave eqn. is some function $Q[u]$ depending on $u(x,t)$, a soln., such that

$$\frac{dQ[u]}{dt} = 0.$$

[2 marks]
[Bookwork].

d) Given
$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & x \in [0, L],\ t \geq 0. \\ u(x,0) = f(x), \\ u_t(x,0) = g(x), \\ u_x(0,t) = u_x(L,t) = 0. \end{cases}$$

show that $\int_0^L (u_t^2 + c^2 u_x^2)\,dx$ is conserved.

↳ Compute

passing d/dt in the integral

$$\frac{d}{dt}\int_0^L (u_t^2 + c^2 u_x^2)\,dx = \int_0^L \left(\frac{\partial}{\partial t}(u_t^2) + c^2 \frac{\partial}{\partial t}(u_x^2) \right) dx$$

$$= 2\int_0^L (u_t u_{tt} + c^2 u_x u_{xt})\,dx$$

Chain rule

[2 marks]

$$= 2\int_0^L u_t u_{tt}\,dx + 2 u_t u_x \Big|_0^L - 2c^2 \int_0^L u_{xx} u_t\,dx$$

integrating by parts in the second term

[2 marks]

wave eqn.

$$= 2 u_t u_x \Big|_0^L + \int_0^L u_t(\underbrace{u_{tt} - c^2 u_{xx}}_{0})\,dx$$

$$= 2\left(u_t(L,t)\underbrace{u_x(L,t)}_{0} - u_t(0,t)\underbrace{u_x(L,t)}_{0}\right)$$

BC's.

[2 marks]

$$= 0.$$

[similar to CW, lectures, partly unseen]

e) Consider the boundary conds.

$$u(0,t) = a, \quad u(L,t) = b.$$

↳ The same computation as before leads to

$$\frac{d}{dt}\int_0^L (u_t^2 + c^2 u_x)\,dx = 2 u_x u_t \Big|_0^L = 0$$

$u(0,t)=a,\ u(L,t)=b \Rightarrow u_t(0,t)=u_t(L,t)=0.$

[3 marks]

[unseen]

Question 3.

a) Explain the difference between

$$(*_1) \quad u(x,t) = \frac{1}{2}\left(f(x+ct)+f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

$$(*_2) \quad u(x,t) = F(x-ct) + G(x+ct).$$

↳ Formula $(*_1)$ is D'Alembert's formula, the unique soln to the initial value problem for the wave eqn. on the real line with conds

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad x \in \mathbb{R}.$$

↝ Formula $(*_2)$ is the general soln. to the wave eqn written in terms of two arb. functs. of a single variable.

[3 marks] [Bookwork, partly unseen]

b) Show that $u(x,t) = \int_{x-ct}^{x+ct} g(s)\,ds$ is a solution to the wave eqn.

$$\rightsquigarrow \quad u(x,t) = \int_0^{x+ct} g(s)\,ds + \int_{x-ct}^{0} g(s)\,ds = \int_0^{x+ct} g(s)\,ds - \int_0^{x-ct} g(s)\,ds$$

[2 marks]

Thus

fund. thm. Calculus

$$u_x(x,t) = g(x+ct) - \cancel{g(0)} - g(x-ct) + \cancel{g(0)}$$

chain rule

$$u_{xx}(x,t) = g'(x+ct) - g'(x-ct)$$

Similarly, FT.C + chain rule

$$u_t(x,t) = cg(x+ct) - \cancel{cg}(0) + cg(x-ct) - \cancel{cg}(0)$$

$$u_{tt}(x,t) = c^2 g'(x+ct) - c^2 g'(x-ct).$$ [2 marks]

Hence, substituting:

$$u_{tt} - c^2 u_{xx} = \cancel{c^2 g'(x+ct)} - \cancel{c^2 g'(x-ct)} - \cancel{c^2 g'(x+ct)} + \cancel{c^2 g'(x-ct)}$$

$$= 0$$

[Partly unseen].

d) Given a solution $u(x,t)$ to

$$u_{tt} - c^2 u_{xx} = 0$$

show that $V(x,t) \equiv u(x+\alpha, t+\beta)$ is also a solution.

↳ Use the chain rule. Let $v = x+\alpha$, $w = t+\beta$.

Thus $V(x,t) = u(v,w)$. One thus gets

$$\begin{cases} V_x = u_v \\ V_{xx} = u_{vv} \end{cases} \qquad \begin{cases} V_t = u_w \\ V_{tt} = u_{ww} \end{cases}$$ [2 marks]

Thus, $V_{tt} - c^2 V_{xx} = \underbrace{u_{vv} - c^2 u_{ww}}_{\text{wave eqn in } (v,w).} = 0$. [1 mark]

Interpretation: the eqn. is invariant under time and space translations. [1 mark]

[Similar to CW/ lectures]

d) Find the solution to the problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \quad , \quad x \in \mathbb{R} \\ u(x,0) = 0, \\ u_t(x,0) = \dfrac{1}{1+x^2} \end{cases}$$

Provide a sketch of the solution.

↳ Use D'Alembert's formula. In this case

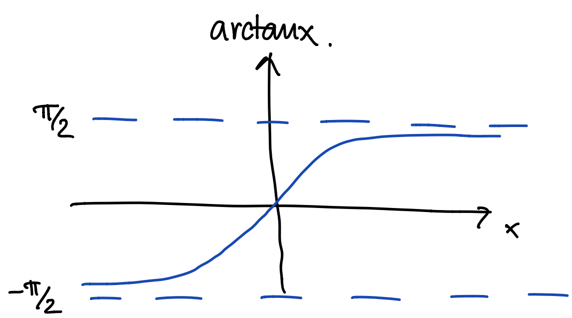
$$f(x) = 0, \quad g(x) = \frac{1}{1+x^2}$$

Recall that (appendix): $\displaystyle\int \frac{dx}{1+x^2} = \arctan x$.

$$\Rightarrow u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \frac{ds}{1+s^2} = \frac{1}{2c}\left(\arctan(x+ct) - \arctan(x-ct)\right).$$

⇒ Observe that $u(x,0) = 0$. [2 marks]

↳ Sketch:

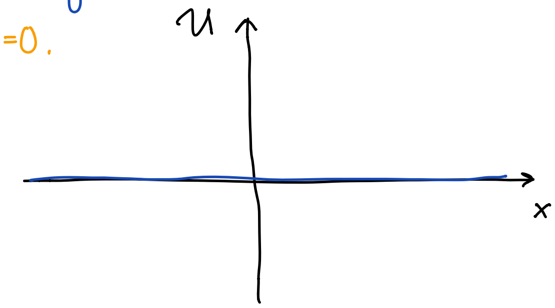


$\arctan(x+ct) \rightsquigarrow$ shifted to the left for $t>0$.

$\arctan(x-ct) \rightsquigarrow$ shifted to the right for $t>0$.

Superimposing:

t=0.



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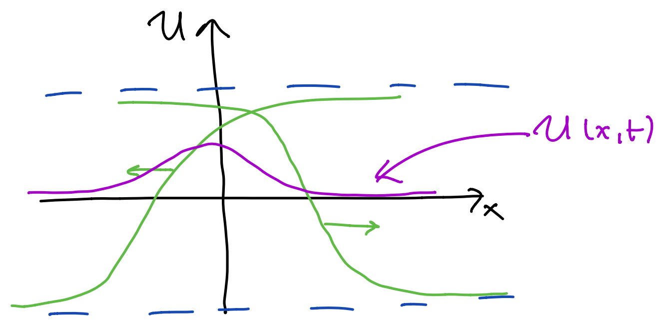

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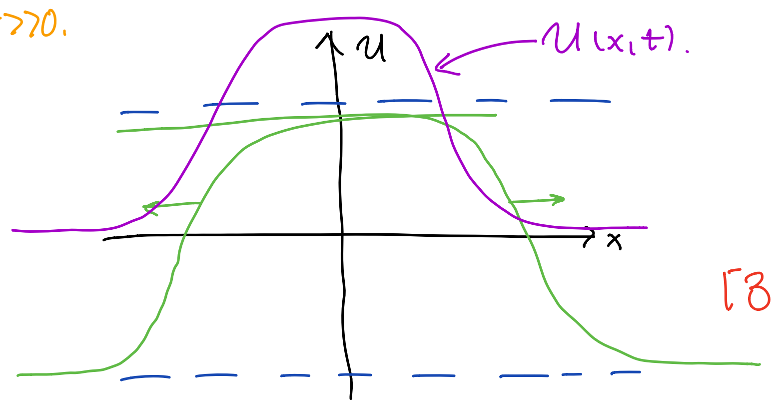


[3 marks]

[similar to cw/lectures, partly unseen].

Question 4.

Consider the problem:

$$\Delta u = 0, \quad (r,\theta) \in \Omega = \{ a \leq r \leq b,\ \theta \in [0, 2\pi) \},$$

$$u(a,\theta) = f(\theta), \quad u(b,\theta) = g(\theta).$$

a) What does the principle of the maximum says if

$$f(\theta) = 1, \quad g(\theta) = 2.$$

↳ As a consequence of the principle of the maximum, the solution has maximum value $u(r,\theta) = 2$ at $r = b$ and minimum value $u(r,\theta) = 1$ at $r = a$.

[3 marks] [similar to cw/lectures]

b) What happens if $f(\theta) = g(\theta) = 1$.

↳ In this case the solution has the same constant value throughout the boundary. Thus, the solution is constant with value 1 on the whole of the annular region.

[3 marks] [similar to cw/lectures].

c) Let $u(r,\theta) = R(r)\Theta(\theta)$. Find the solutions satisfied by R and Θ.

↳ $$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \quad \text{(Appendix)}$$

$$\Rightarrow R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0.$$

Multiplying by $\frac{r^2}{R\Theta}$: $\frac{r^2 R''}{R} + \frac{rR'}{R} + \frac{\Theta''}{\Theta} = 0$

$$\Rightarrow \underbrace{\frac{r^2 R''}{R} + \frac{rR'}{R}}_{\text{depends only on } r} = \underbrace{-\frac{\Theta''}{\Theta}}_{\text{depends only on } \theta.} = k$$

separation constant.
[2 marks]

$$\therefore \begin{cases} r^2 R'' + rR' - kR = 0, & (*_1) \\ \Theta'' + k\Theta = 0. & (*_2). \end{cases}$$

[2 marks]

[similar to lectures].

d) Periodic solutions?

The eqn $(*_2)$ has periodic solutions if $k \geq 0$. In this case

$\Theta(\theta) = c_1 \cos(\sqrt{k}\,\theta) + c_2 \sin(\sqrt{k}\,\theta)$. $k \neq 0$

$\Theta = c_1 \theta + c_2 \qquad k = 0$

[2 marks]

[lectures]

[2 marks]

e) $k = 0$.

f) Find the solns to $(*_1)$ and $(*_2)$ if $k = 0$.

If $k = 0$ then $\begin{cases} r^2 R'' + rR' = 0 & (*_1') \\ \Theta'' = 0 & (*_2'). \end{cases}$

The only periodic solution to $(*_2')$ is $\Theta(\theta) = \text{constant}$

(w.l.g. set $\Theta(\theta) = 1$).

↳ If $r \neq 0$ then $(*_1')$ yields $rR'' = -R'$. [2 marks]

$$\Rightarrow r\frac{dR'}{dr} = -R' \rightsquigarrow \int \frac{dR'}{R'} = -\int \frac{dr}{r} + C_1$$

$$\Rightarrow \ln R' = -\ln r + \ln C_1$$ ~ redefining C_1.

$$\therefore \; R' = \frac{C_1}{r}.$$

Integrating one last time:

$$R(r) = C_1 \ln r + C_2, \quad C_1, C_2 \in \mathbb{R}$$ [3 marks]

[similar to CW / lectures].

g) Use the above to solve

$$\Delta u = 0, \quad (r,\theta) \in \Omega = \{ a \leq r \leq b, \; \theta \in [0, 2\pi) \}$$

$$u(a,\theta) = 1, \quad u(b,\theta) = 2.$$

From the above the solution which is constant at the boundary is

$$u(r,\theta) = \Theta(\theta) R(r)$$
$$= C_1 \ln r + C_2.$$ [2 marks]

↳ Fix C_1 and C_2 via boundary conds:

$$\begin{cases} u(a,\theta) = C_1 \ln a + C_2 = 1 \\ u(b,\theta) = C_2 \ln b + C_2 = 2 \end{cases}$$

Solving the linear system one gets:

$$C_1 = \frac{1}{\ln a - \ln b} = \frac{1}{\ln a/b}$$

$$C_2 = \frac{2\ln a - \ln b}{\ln a - \ln b} = \frac{\ln a^2/b}{\ln a/b}$$ [3 marks]

[Partially unseen].

Hence,

$$u(r,\theta) = \frac{1}{\ln a/b}\ln r + \frac{\ln a^2/b}{\ln a/b}.$$

(h) <u>Uniqueness.</u>

$\hookrightarrow$ Let u_1 and u_2 be solutions with the same boundary conds.
Moreover, let $V \equiv u_2 - u_1$. By linearity one has

$$\Delta u = 0 \quad \text{with} \quad u\big|_{\partial\Omega} = 0.$$

Now, the solution is constant on $\partial\Omega$, so that u is constant throughout Ω. Hence $V=0$ on Ω and $u_1 = u_2$ on Ω $\blacksquare$

[4 marks]

[Similar to lectures].

Question 5.

(a) Given
$$\begin{cases} X''(x) = kX(x) \\ X(-a) = X(a), \quad X'(-a) = X'(a) \end{cases}$$

show that $k<0$.

↳ Starting from $X''(x) = kX(x)$ multiply by $X(x)$ and integrate over $[-a,a]$:

$$\int_{-a}^{a} X X'' \,dx = k \int_{-a}^{a} X^2 \,dx$$ [2 marks]

Now using integration by parts on the left hand side:

$$X X' \Big|_{-a}^{a} - \int X'^2 \,dx = k \int_{-a}^{a} X^2 \,dx$$

BC's

$$\Rightarrow \cancel{X(a)X'(a)} - \cancel{X(-a)X'(-a)} - \int X'^2 dx = k \int_{-a}^{a} X^2 dx$$

$$\therefore -\underbrace{\int_{-a}^{a} X'^2 \,dx}_{k \geq 0} = k \underbrace{\int_{-a}^{a} X^2 dx}_{k \geq 0.}$$ [3 marks].

$\therefore k<0$ ■

[Similar to HW / lectures]

b) Why is the original problem for the heat eqn?

One has periodic boundary conditions. Thus,

$$\begin{cases} u_t = \varkappa u_{xx} & x \in [-a,a] \\ u(-a,0) = u(a,0) \\ u_x(-a,0) = u_x(a,0). \end{cases}$$

[3 marks]

[Unseen in this form].

(c) Use the Fourier-Poisson formula

$$u(x,t) = \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{4\varkappa t}}}{\sqrt{4\varkappa \pi t}} f(y)\,dy$$

to compute the solution to

$$\begin{cases} u_t = \varkappa u_{xx}, & x \in \mathbb{R}, \quad t > 0. \\ u(x,0) = 1. \end{cases}$$

Provide an interpretation.

↳ Substitution of the initial condition on the Fourier formula gives

$$u(x,t) = \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{4\varkappa t}}}{\sqrt{4\varkappa \pi t}}\,dy$$

[2 marks]

$$= \frac{1}{\sqrt{4\varkappa \pi t}} \int_{-\infty}^{\infty} e^{-s^2} \sqrt{4\varkappa t}\,ds = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-s^2} ds = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1.$$

$s = (y-x)/\sqrt{4\varkappa t}$

$ds = dy/\sqrt{4\varkappa t}$

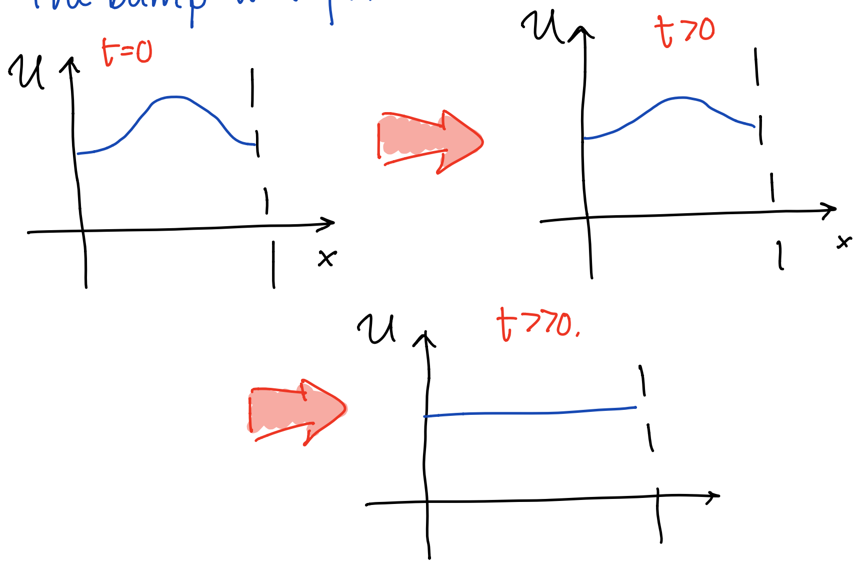
[3 marks].

$\therefore\ u(x,t) = 1.$

↳ One has an initial constant distribution of temperature. Thus, it must remain constant!

[similar to cw/lectures].

d) The bump will flatten as $t \to \infty$.



[3 marks]

↳ As the temperature is kept at the fixed value of 1 at the extremes, it therefore cannot go below 1. Thus, one expects that $u(x,t) \longrightarrow 1$ as $t \longrightarrow \infty$.

[2 marks]

[similar to cw/lectures].