Queen Mary University of London

Main Examination period 2023 – January – Semester A

MTH6151: Main Exam Solutions

Comments on questions:

(1) Question 1 (a),(d) are test of concepts understanding based on the lectures. (b) is similar to questions in weekly problem sets, (c) is solving PDE based on ODE methods, if the student's figure out which method to use they should be able to solve.
(2) Question 2 (a) is similar to questions in the weekly problems. 2(b) is a new format asking students to judge true or false, testing about scaling properties of various PDEs.
(3) Question 3(a) is students' ability to apply maximum principle. 3(b) is a variation of a coursework problem.
(4) Question 4(a) is new, for good students that know how to apply the right formulas and theorems for Laplace equations to the domain of an annulus.
(5) Question 5(a) is similar to questions in previous exams. 5(b) is an application of maximum principle for heat equation 5(c) is similar to previous exams 5(d) is test of concepts understanding.

Question 1 [29 marks].

(a) For each of the following equations, write down the order of the equation, determine whether each of them is linear or non-linear, and say whether they are homogeneous or inhomogeneous.

(1) $e^y U_{xxy} + e^x U_{yyx} + x^4 U = 0$.

(2) $U^2 \cdot \Delta U + \Delta(U_x) + 3U_y = 2023$. **[6]**

Solution:

(1) This is a 3rd order, linear homogeneous equation. 3 marks

(2) This is a 3rd order, non-linear, inhomogeneous equation. 3 marks

One mark for each of the judgement on order, linearity and homogeneity

(b) Consider the equation $U_x + tU_t = -1$.

(1) Find the characteristics of this equation.

(2) Find the general solutions to this equation.

(3) Solve the following boundary value problem for this equation

$$\begin{cases} U_x + tU_t = -1 \\ U(0,t) = t. \end{cases}$$

[10]

Solution:

(1) The characteristics are given by solutions to the ODE $\frac{\partial t}{\partial x} = t$. i.e. $t = Ce^x$. (2 marks)

(2) Along the characteristics, the equation becomes

$$\begin{aligned} \frac{d}{dx}U &= -1 \\ U(x, t(x)) &= -x + f(C). \end{aligned}$$

Using $C = te^{-x}$, we have the general solutions

$$U(x,t) = -x + f(te^{-x}).$$

(4 marks)

(3) When $x = 0$, we have $t = C$ and thus

$$C = t = U(0,t) = -0 + f(C).$$

So $f(C) = C$ and the solution to the boundary value problem is

$$U(x,t) = -x + te^{-x}.$$

(4 marks)

(c) Find the general solutions $U(x,t)$ to the equation

$$U_t + U_{xt} = 0$$

[8]

Solution:

Integrate both sides with respect to t (2 marks), we get

$$U + U_x = \tilde{f}(x).$$

(2 marks)

Using integrating factor e^x we get

$$\begin{aligned} e^x U + e^x U_x &= e^x \tilde{f}(x) \\ \frac{\partial}{\partial x}(e^x U) &= f(x) \\ e^x U(x,t) &= F(x) + g(t) \\ U(x,t) &= \tilde{F}(x) + e^{-x} g(t), \end{aligned}$$

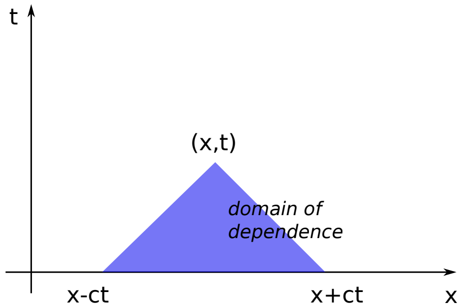
for arbitrary $\tilde{F}$ and g. (4 marks)

2 marks each for obtaining the terms $\tilde{F}(x)$ and $e^{-x}g(t)$

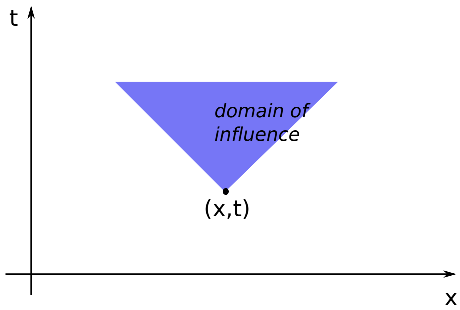
(d) Describe the meaning of domain of dependence and domain of influence, and then interpret how the solutions of wave equations are influenced by the initial condition using D'Alembert's formula. [5]

Solution:

For any $(x,t), x \in \mathbb{R}, t \geq 0$, the domain of dependence is



and the domain of influence is



(2 marks)

This tells that the information can only travel with a finite speed c. (1 mark)

D'Alembert's formula tells that the initial data influence the solutions of wave equation in the following manner.

$$U(x,t) = \big(\text{average of } U(x,0) \text{ on } x-ct \text{ and } x+ct\big) \\ +\big(\text{average of } U_t(x,0) \text{ over the interval } [x-ct, x+ct]\big).$$

(2 marks)

Question 2 [19 marks].

(a) Write down the principal part of the equation $-U + U_x - U_y - U_{xy} + U_{yy} = 2x$, and then determine the type (elliptic, parabolic or hyperbolic) of this equation. [**3**]

Solution:

The principal part is $-U_{xy} + U_{yy}$.

With $a = 0, b = \frac{1}{2}, c = 1$, we have $b^2 - ac = \frac{1}{4} > 0$. So the equation is hyperbolic. (3 marks)

(b) Decide whether the following statements are true or false. (You don't need to explain your answer)

(1) If $U(x,t)$ is a solution to the wave equation $U_{tt} - c^2 U_{xx} = 0$, then $V(x,t) = U(2x, -2t)$ is also a solution to the same wave equation.

(2) If $U(x,y)$ is a harmonic function, then $V(x,y) = [U(x,y)]^3$ is also harmonic.

(3) If $U(x,t)$ is a solution to the heat equation $U_t - \varkappa U_{xx} = 0$, then $V(x,t) = U(x, -t)$ is also a solution to the same heat equation.

(4) If $U(x,t)$ is a solution to the heat equation $U_t - \varkappa U_{xx} = 0$ and f is a compactly supported differentiable function defined on $\mathbb{R}$, then the function $V(x,t)$ defined by the convolution $V(x,t) = \int_{-\infty}^{\infty} U_t(x-y,t) f(y) dy$ is also a solution to the same heat equation. (Here U_t is the partial derivative of U with respect to t.) [8]

Solution:

(1) True

(2) False

(3) False

(4) True

(8 marks) 2 marks for each judgement

(c) Consider the eigenvalue problem

$$\begin{cases} X'' = -\lambda X, x \in [0,3] \\ X(0) = 0, X(3) = 0. \end{cases}$$

(1) Show that the eigenvalues λ are all positive.

(2) Compute all the eigenvalues. [8]

Solution:

(1) Multiply both sides of the equation by X and integrate from 0 to 3 we get

$$\int_0^3 X(x)X''(x)dx = \int_0^3 -\lambda X^2(x)dx.$$

Using integration by parts, we have

$$\begin{aligned} X(x)X'(x)|_0^3 - \int_0^3 (X'(x))^2 dx &= -\lambda \int_0^3 X^2(x)dx \\ - \int_0^3 (X'(x))^2 dx &= -\lambda \int_0^3 X^2(x)dx \end{aligned}$$

where we used the boundary conditions. Now for any X not constantly zero, we have $\int_0^3 (X'(x))^2 dx > 0$ and $\int_0^3 X^2(x)dx > 0$. So we conclude that $\lambda > 0$.

(4 marks)

(2) By the theory for 2nd order linear ODEs and the fact that λ is positive, the characteristic polynomial is $x^2 + \lambda = 0$ has 2 complex roots $-i\sqrt{\lambda}$ and $o\sqrt{\lambda}$. So the general solutions are

$$X(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x).$$

The first boundary condition $X(0) = 0$ implies $0 = X(0) = C_1 + 0$. So $C_1 = 0$ and thus $C_2 \neq 0$.

The second boundary condition $X(3) = 0$ then implies
$0 = X(3) = 0 + C_2 \sin(3\sqrt{\lambda})$. So $3\sqrt{\lambda} = n\pi$ for $n = 1, 2, \ldots$ and thus the eigenvalues are

$$\lambda_n = \frac{n^2\pi^2}{9}.$$

(4 marks)

Question 3 [16 marks].

(a) Solve the following inhomogeneous wave equation on the real line

$$\begin{cases} U_{tt} - c^2 U_{xx} = 2x - \sin x \\ U(x,0) = \cos^2 x, U_t(x,0) = 1. \end{cases}$$

[**8**]

Solution: Using Duhamel's principle, the solutions for the inhomogeneous wave equation on the real line is

$$\begin{aligned} U(x,t) =& \frac{1}{2}[\cos^2(x+ct) + \cos^2(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} 1 ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} (2r - \sin r) dr ds \\ =& \frac{1}{2}[\frac{1-\cos(2(x+ct))}{2} + \frac{1-\cos(2(x-ct))}{2}] + \frac{2ct}{2c} \\ &+ \frac{1}{2c}\int_0^t [r^2 + \cos r]|_{x-c(t-s)}^{x+c(t-s)} ds (4 marks) \\ =& \frac{1}{2} + \frac{\cos(2(x+ct)) + \cos(2(x-ct))}{4} + t \\ &+ \frac{1}{2c}\int_0^t [4cx(t-s) + \cos(x+ct-cs) - \cos(x-ct+cs)] ds \\ =& \frac{1}{2} + \frac{\cos(2(x+ct)) + \cos(2(x-ct))}{4} + t + xt^2 \\ &- \frac{\sin x}{c^2} + \frac{\sin(x+ct) + \sin(x-ct)}{2c^2}. (4 marks) \end{aligned}$$

(b) (1) Suppose $U(x,t)$ is compactly supported for all time and is a solution to the hyperbolic equation

$$U_{tt} - 4U_{xx} + 2U_t = 0, x \in \mathbb{R}.$$

Show that the energy $E[U](t) = \frac{1}{2}\int_{-\infty}^{\infty}(U_t^2 + 4U_x^2)dx$ is non-increasing in time.

(2) Use the above fact about energy non-increasing in time to show that if the solution to the following initial value problem exists then it must be unique.

$$\begin{cases} U_{tt} - 4U_{xx} + 2U_t = \psi(x), x \in \mathbb{R} \\ U(x,0) = f(x), U_t(x,0) = g(x). \end{cases}$$

[**8**]

Solution:

(1) We compute

$$\begin{aligned}
\frac{d}{dt}E[U](t) =& \frac{d}{dt}\frac{1}{2}\int_{-\infty}^{\infty}(U_t^2 + 4U_x^2)dx \\
=& \frac{1}{2}\int_{-\infty}^{\infty}\frac{d}{dt}[U_t^2 + 4U_x^2]dx \\
=& \frac{1}{2}\int_{-\infty}^{\infty}[2U_tU_{tt} + 8U_xU_{tx}]dx \\
=& \int_{-\infty}^{\infty}U_tU_{tt}dx + 4\int_{-\infty}^{\infty}U_xU_{tx}dx \\
=& \int_{-\infty}^{\infty}U_tU_{tt}dx + 4U_xU_t|_{-\infty}^{\infty} - 4\int_{-\infty}^{\infty}U_{xx}U_tdx \\
=& \int_{-\infty}^{\infty}U_tU_{tt}dx + 0 - 4\int_{-\infty}^{\infty}U_{xx}U_tdx \\
=& 4\int_{-\infty}^{\infty}U_t(U_{tt} - 4U_{xx})dx \\
=& 4\int_{-\infty}^{\infty}-2U_t^2dx \\
\leq& 0,
\end{aligned}$$

where we used the integration by parts in the fifth line and the compactness of support in the sixth line.

So the energy is non-increasing. (4 marks)

(2) Suppose U_1 and U_2 are 2 solutions to this initial value inhomogeneous problem. Then by the principle of superposition, we know that $U = U_1 - U_2$ is a solution to the homogeneous equation with zero initial value, i.e.

$$\begin{cases} U_{tt} - 4U_{xx} + 2U_t = 0, x \in \mathbb{R} \\ U(x,0) = 0, U_t(x,0) = 0. \end{cases}$$

By the zero initial value, we see that $E[U](0) = 0$. Because $E[U](t) \geq 0$ and is non-increasing, we know that $E[U](t) \equiv 0$ for all t, this forces $U(x,t) \equiv 0$ and thus $U_1 = U_2$.

(4 marks)

Question 4 [16 marks].

(a) (1) Find the solution $U(r,\theta)$ to the Laplace equation in the annulus $1 \leq r \leq 2$ with the boundary conditions

$$\begin{cases} U(1,\theta) = 3\cos\theta - 1 \\ U(2,\theta) = 3\cos\theta - 1. \end{cases}$$

(2) Show that the solution U obtained above satisfies $U \leq 2$ and $U \geq -4$ in the whole annulus. **[11]**

Solution:

(1) The general solutions for Laplace equation in polar coordinate is

$$U(r,\theta) = \Big(C_0 + D_0 \ln r\Big) + \sum_{m=1}^{\infty} \Big(C_m r^m + \frac{D_m}{r^m}\Big)(A_m \cos m\theta + B_m \sin m\theta).$$

(3 marks) By the boundary conditions, we see that $C_m, D_m, B_m = 0$ for all $m \geq 2$, and

$$C_0 + D_0 \ln 1 + (C_1 + D_1)A_1 \cos\theta = 3\cos\theta - 1$$
$$C_0 + D_0 \ln 2 + (2C_1 + \frac{D_1}{2})A_1 \cos\theta = 3\cos\theta - 1.$$

(3 marks) By observation one can choose $C_0 = -1, D_0 = 0, C_1 = 1, D_1 = 2, A_1 = 1$. So plugging into the general solutions, we get the solution to the boundary value problem is

$$U(r,\theta) = -1 + (r + \frac{2}{r})\cos\theta.$$

(2 marks)

(2) Using that $-1 \leq \cos\theta \leq 1$, we have $-4 \leq U(1,\theta) \leq 2$ and $-4 \leq U(2,\theta) \leq 2$ on the 2 boundaries of the annulus. So by maximum principle, we conclude that $2 \leq U \leq 4$ in the whole annulus. (3 marks)

(b) Suppose that U is a harmonic function in the disk $\Omega = \{r < 3\}$ and that

$$U(3,\theta) = \sin\theta + \cos 2\theta.$$

Without finding the solution, compute the value of U at the origin – that is, at $r = 0$. **[5]**

Solution:
By the first mean value theorem (2 marks), the value at the origin should be equal to the average on the circle of radius 3, which is equal to
$\frac{1}{2\pi}\int_0^{2\pi} U(3,\theta)d\theta = \frac{1}{2\pi}\int_0^{2\pi}(\sin\theta + \cos 2\theta)d\theta = 0$. (3 marks)

Question 5 [20 marks].

(a) Determine all possible values of a, b, c so that $U(x,t) = ax + bx^2 + ct$ is a solution to the heat equation $U_t - \varkappa U_{xx} = 0$. **[6]**

Solutions:

The equation

$$U_t - \varkappa U_{xx} = 0.$$

implies

$$c - 2b\varkappa = 0.$$

So a can be any real number. (2 marks) And b, c satisfies $c = 2b\varkappa$. (4 marks)

(b) Consider the following initial and boundary value problem to the heat equation

$$U_t - \varkappa U_{xx} = 0, -2\pi \le x \le 2\pi, t > 0$$
$$U(-2\pi, 0) = 1, U(2\pi, 0) = 3$$
$$U(x,0) = \begin{cases} 2 + \cos x, \ \pi \le x \le 2\pi \\ 1, \ -\pi < x < \pi \\ 1 - \sin x, \ -2\pi \le x \le -\pi. \end{cases}$$

Without solving the equation, show that $U(x,t) \ge 0$ and $U(x,t) \le 3$ for all $x \in \mathbb{R}, 0 < t < 1$. **[5]**

Solutions: Using that $-1 \le \cos x, \sin x \le 1$, we have that $2 + \cos x \in [1,3]$ and $1 - \sin x \in [0,2]$. (2 marks) So we get $0 \le U \le 3$ on the boundary of $\Omega = \{-2\pi \le x \le 2\pi, 0 < t < 1\}$. By the maximum principle, we conclude that $U \ge 0$ and $U \le 3$ on the whole Ω. (3 marks)

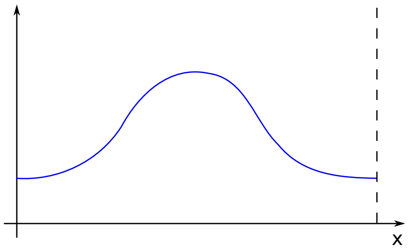
(c) Describe in qualitative terms the behaviour of the solution to the heat equation on an interval

$$U_t = \varkappa U_{xx}, \qquad x \in [0, 2\pi],$$

with initial data

$$U(x,0) = f(x)$$

where $f(x)$ has the form

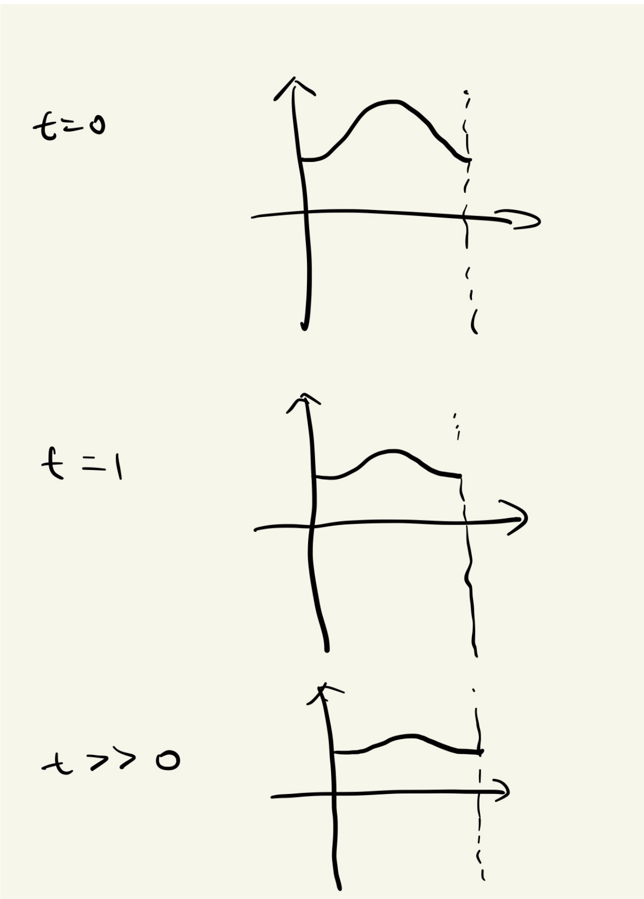


and

$$U(0,t) = U(2\pi,t) = 1.$$

What do you expect to be the limit of $U(x,t)$ as $t \to \infty$? No proof or calculations are required. You may draw a plot of the solution at various instants of time to explain your answer. [**5**]

Solution: The limit behavior of U is that it's becoming flatter and flatterA sketch is as below:



(3 marks)

As time go to infinity, the value of U tend to 1. (2 marks)

(d) Describe in words (with a maximum 4 sentenses) the procedure of solving heat equations on the half-line with Dirichlet boundary conditions.

$$U_t = \varkappa U_{xx}, x \geq 0, t > 0$$
$$U(x, 0) = f(x)$$
$$U(0, t) = 0.$$

[**4**]

Solution:

Step 1: Do an odd extension of f to the whole real line and get F.(1 mark)

Step 2: Solve the initial value problem on the real line

$$V_t = \varkappa V_{xx}, x \in \mathbb{R}, t > 0$$
$$V(x, 0) = F(x)$$

using Fourier-Poisson formula.(2 marks)

Step 3: The restriction of V on the half line is the solution U of the heat equation on half-line with Dirichlet condition. (1 mark)

End of Paper.