Queen Mary
University of London

Main Examination period 2023 – January – Semester A

MTH6151 / MTH6151P: Partial Differential Equations

Duration: 2 hours

The exam is intended to be completed within **2 hours**. However, you will have a period of **4 hours** to complete the exam and submit your solutions.

You should attempt ALL questions. Marks available are shown next to the questions.

All work should be **handwritten** and should **include your student number**. Only one attempt is allowed – **once you have submitted your work, it is final**.

In completing this assessment:

- You may use books and notes.
- You may use calculators and computers, but you must show your working for any calculations you do.
- You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.
- You must not seek or obtain help from anyone else.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;

Examiners: S. Wang, O. Jenkinson

Question 1 [29 marks].

(a) For each of the following equations, write down the order of the equation, determine whether each of them is linear or non-linear, and say whether they are homogeneous or inhomogeneous.

(1) $e^y U_{xxy} + e^x U_{yyx} + x^4 U = 0$.

(2) $U^2 \cdot \Delta U + \Delta(U_x) + 3U_y = 2023$. [**6**]

(b) Consider the equation $U_x + tU_t = -1$.

(1) Find the characteristics of this equation.

(2) Find the general solutions to this equation.

(3) Solve the following boundary value problem for this equation

$$\begin{cases} U_x + tU_t = -U + 1 \\ U(0,t) = t. \end{cases}$$

[**10**]

(c) Find the general solutions $U(x,t)$ to the equation

$$U_t + U_{xt} = 0.$$

[**8**]

(d) Describe the meaning of domain of dependence and domain of influence, and then interpret how the solutions of wave equations are influenced by the initial condition using D'Alembert's formula. [**5**]

Question 2 [19 marks].

(a) Write down the principal part of the equation $-U + U_x - U_y - U_{xy} + U_{yy} = 2x$, and then determine the type (elliptic, parabolic or hyperbolic) of this equation.

[**3**]

(b) Decide whether the following statements are true or false. (You don't need to explain your answer)

(1) If $U(x,t)$ is a solution to the wave equation $U_{tt} - c^2 U_{xx} = 0$, then $V(x,t) = U(2x, -2t)$ is also a solution to the same wave equation.

(2) If $U(x,y)$ is a harmonic function, then $V(x,y) = [U(x,y)]^3$ is also harmonic.

(3) If $U(x,t)$ is a solution to the heat equation $U_t - \varkappa U_{xx} = 0$, then $V(x,t) = U(x, -t)$ is also a solution to the same heat equation.

(4) If $U(x,t)$ is a solution to the heat equation $U_t - \varkappa U_{xx} = 0$ and f is a compactly supported differentiable function defined on $\mathbb{R}$, then the function $V(x,t)$ defined by the convolution $V(x,t) = \int_{-\infty}^{\infty} U_t(x-y,t) f(y) dy$ is also a solution to the same heat equation. (Here U_t is the partial derivative of U with respect to t.) [**8**]

(c) Consider the eigenvalue problem

$$\begin{cases} X'' = -\lambda X, x \in [0,3] \\ X(0) = 0, X(3) = 0. \end{cases}$$

(1) Show that the eigenvalues λ are all positive.

(2) Compute all the eigenvalues. [**8**]

Question 3 [16 marks].

(a) Solve the following inhomogeneous wave equation on the real line

$$\begin{cases} U_{tt} - c^2 U_{xx} = 2x - \sin x \\ U(x,0) = \cos^2 x, U_t(x,0) = 1. \end{cases}$$

[8]

(b) (1) Suppose $U(x,t)$ is compactly supported for all time and is a solution to the hyperbolic equation

$$U_{tt} - 4U_{xx} + 2U_t = 0, x \in \mathbb{R}.$$

Show that the energy $E[U](t) = \frac{1}{2}\int_{-\infty}^{\infty}(U_t^2 + 4U_x^2)dx$ is non-increasing in time.

(2) Use the above fact about energy non-increasing in time to show that if the solution to the following initial value problem exists then it must be unique.

$$\begin{cases} U_{tt} - 4U_{xx} + 2U_t = \psi(x), x \in \mathbb{R} \\ U(x,0) = f(x), U_t(x,0) = g(x). \end{cases}$$

[8]

Question 4 [16 marks].

(a) (1) Find the solution $U(r,\theta)$ to the Laplace equation in the annulus $1 \leq r \leq 2$ with the boundary conditions

$$\begin{cases} U(1,\theta) = 3\cos\theta - 1 \\ U(2,\theta) = 3\cos\theta - 1. \end{cases}$$

(2) Show that the solution U obtained above satisfies $U \leq 2$ and $U \geq -4$ in the whole annulus. **[11]**

(b) Suppose that U is a harmonic function in the disk $\Omega = \{r < 3\}$ and that

$$U(3,\theta) = \sin\theta + \cos 2\theta.$$

Without finding the solution, compute the value of U at the origin – that is, at $r = 0$. **[5]**

Question 5 [20 marks].

(a) Determine all possible values of a, b, c so that $U(x,t) = ax + bx^2 + ct$ is a solution to the heat equation $U_t - \varkappa U_{xx} = 0$. **[6]**

(b) Consider the following initial and boundary value problem to the heat equation

$$
\begin{aligned}
&U_t - \varkappa U_{xx} = 0, -2\pi \le x \le 2\pi, t > 0 \\
&U(-2\pi, 0) = 1, U(2\pi, 0) = 3 \\
&U(x,0) = \begin{cases} 2 + \cos x, \ \pi \le x \le 2\pi \\ 1, \ -\pi < x < \pi \\ 1 - \sin x, \ -2\pi \le x \le -\pi. \end{cases}
\end{aligned}
$$

Without solving the equation, show that $U(x,t) \ge 0$ and $U(x,t) \le 3$ for all $x \in \mathbb{R}, 0 < t < 1$. **[5]**

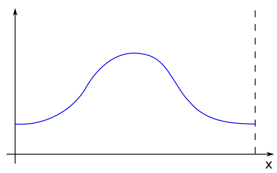
(c) Describe in qualitative terms the behaviour of the solution to the heat equation on an interval

$$U_t = \varkappa U_{xx}, \qquad x \in [0, 2\pi],$$

with initial data

$$U(x,0) = f(x)$$

where $f(x)$ has the form



and

$$U(0,t) = U(2\pi, t) = 1.$$

What do you expect to be the limit of $U(x,t)$ as $t \to \infty$? No proof or calculations are required. You may draw a plot of the solution at various instants of time to explain your answer. **[5]**

(d) Describe in words (with a maximum 4 sentences) the procedure of solving heat equations on the half-line with Dirichlet boundary conditions:

$$
\begin{aligned}
&U_t = \varkappa U_{xx}, x \ge 0, t > 0 \\
&U(x,0) = f(x) \\
&U(0,t) = 0.
\end{aligned}
$$

[4]

End of Paper.